The $\star$-operator and Invariant Subtraction Games

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Abstract

An *invariant subtraction game* is a 2-player impartial game defined by a set of invariant moves (k-tuples of non-negative integers) $\mathcal{M}$. Given a position (another k-tuple) $\boldsymbol{x} = (x_1, \ldots, x_k)$, each option is of the form $(x_1 - m_1, \ldots, x_k - m_k)$, where $\boldsymbol{m} = (m_1, \ldots, m_k) \in \mathcal{M}$, and where $x_i - m_i \geq 0$, for all i. Two players alternate in moving and the player who moves last wins. The set of non-zero P-positions of the game $\mathcal{M}$ defines the moves in the dual game $\mathcal{M}^\star$. For example, in the game of (2-pile Nim)* a move consists in removing the same positive number of tokens from both piles. Our main results concern a double application of $\star$, the operation $\mathcal{M} \to (\mathcal{M}^\star)^\star$. We establish a fundamental 'convergence' result for this operation. Then, we give necessary and sufficient conditions for the relation $\mathcal{M} = (\mathcal{M}^\star)^\star$ to hold, as is the case for example with $\mathcal{M} = k$-pile Nim.

Keywords: Dual game; Game convergence; Game reflexivity; Impartial game; Invariant subtraction game; $\star$-operator

1 Introduction and terminology

An *invariant subtraction game* [DR10, LHF11] is a two-player *impartial combinatorial game* (see [BCG01] for a background on such games) defined on a set of *positions* represented as k-tuples $\boldsymbol{x} = (x_1, \ldots, x_k)$, where $k \in \mathbb{N} = \{1, 2, \ldots\}$ and $x_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The move options are determined by a set, $\mathcal{M} \subset \mathbb{N}_0^k \setminus \{\mathbf{0}\}$, of *invariant moves*. Each *option*, from a given position $\boldsymbol{x} = (x_1, \ldots, x_k)$, is of the form

$$\boldsymbol{x} \ominus \boldsymbol{m} = (x_1 - m_1, \ldots, x_k - m_k),$$

where $\boldsymbol{m} = (m_1 \ldots, m_k) \in \mathcal{M}$ and where $x_i \geq m_i$, for all i. The latter relation is also denoted $\boldsymbol{x} \succeq \boldsymbol{m}$ (and $\succ$ means that strict inequality holds for at least one coordinate). The players alternate in moving and a player who cannot move loses. Clearly, this setting excludes the possibility of a draw game, but it includes many classical "take-away" games [G66, S70, Z96] played on a finite number of tokens, e.g. Nim [B1902], Wythoff Nim [W1907], the (one-pile) subtraction games in [BCG01].

Remark 1. *Our setting is very similar to the "take-away" games in [G66]. However, since nowadays the term "take-away" often includes the possibility of a certain form of "move dependence" [S70, Z96] which we are not considering here, we prefer to use the terminology introduced in [DR10]. Also, we differ from [G66] in the definition of the ending condition of a game. Golomb's unique winning condition is a move to* $\mathbf{0}$*, so that in his setting many games are draw. (He also allows for the possibility of the vector* $\mathbf{0}$ *as a move.)*

We identify an invariant subtraction game with its set of moves $\mathcal{M}$ and call a position N if the player about to move (the next player) wins; otherwise it is P (the previous player wins). Hence, a position is P if and only if each of its options is N. A position $\boldsymbol{x}$ is *terminal* if $\mathbf{0} \preceq \boldsymbol{y} \preceq \boldsymbol{x}$ implies $\boldsymbol{y} \notin \mathcal{M}$. Hence, each terminal position is P. Altogether this gives that the sets of N- and P-positions are recursively defined. We denote these sets by $\mathcal{N}(\mathcal{M})$ and $\mathcal{P}(\mathcal{M})$ respectively.

Suppose that $X \subseteq \mathbb{N}_0^k$. Then, we denote by X' the set $X \setminus \{\mathbf{0}\}$. Let $\mathcal{M}$ be an invariant subtraction game. Then the *dual game* of $\mathcal{M}$ is defined by $\mathcal{M}^\star = \mathcal{P}(\mathcal{M})'$ and $\mathcal{M}$ is *reflexive* if $\mathcal{M} = \mathcal{P}(\mathcal{M}^\star)'$ that is if $\mathcal{M} = \mathcal{M}^{\star\star}$, where $\mathcal{M}^{\star\star}$ stands for $(\mathcal{M}^\star)^\star$. Note that $\mathcal{M}^\star$ is reflexive whenever $\mathcal{M}$ is.

A sequence of invariant subtraction games $(\mathcal{M}_i)_{i \in \mathbb{N}_0}$ *converges* if, for all $\boldsymbol{x} \in \mathbb{N}_0^k$, there is an $n_0 = n_0(\boldsymbol{x}) \in \mathbb{N}_0$ such that, for all $n \geq n_0$, for all $\boldsymbol{y} \preceq \boldsymbol{x}$, $\boldsymbol{y} \in \mathcal{M}_n$ if and only if $\boldsymbol{y} \in \mathcal{M}_{n_0}$. If $(\mathcal{M}_i)_{i \in \mathbb{N}_0}$ converges, then we can define the unique 'limit-game' of the sequence, denoted by $\lim_{i \in \mathbb{N}_0} \mathcal{M}_i$. For $i \in \mathbb{N}$, let $\mathcal{M}^i$ denote the game $(\mathcal{M}^{i-1})^\star$ where $\mathcal{M}^0 = \mathcal{M}$ is an invariant subtraction game.

Let us state our two main results, proved in Section 2 and 3 respectively.

Theorem 1. *Let $\mathcal{M}^0 = \mathcal{M}$ denote an invariant subtraction game. Then the sequence $(\mathcal{M}^{2i})_{i\in\mathbb{N}_0}$ converges.*

Let $X \subseteq \mathbb{N}_0^k$. Then we denote by $\mathcal{D}(X)$ the set $\{\boldsymbol{x} \ominus \boldsymbol{y} \succ \boldsymbol{0} \mid \boldsymbol{x}, \boldsymbol{y} \in X\}$.

Theorem 2. *Let $\mathcal{M}$ denote an invariant subtraction game. Then the following items are equivalent,*

(a) $\mathcal{M}$ is reflexive,

(b) $\mathcal{M} = \lim_{i\in\mathbb{N}_0} \mathcal{X}^{2i}$, for some invariant subtraction game $\mathcal{X} = \mathcal{X}^0$,

(c) $\mathcal{D}(\mathcal{M}) \subseteq \mathcal{N}(\mathcal{M})$.

In Example 1 and Figure 1 we demonstrate a simple application of Theorem 2 (c). In Example 2 and Figure 2 we show an example of a game which has a very simple structure, but for which we do not know whether reflexivity holds for any game resulting from a finite number of recursive applications of the $\star$-operator. (Due to computer simulations there appears to be many such games.) In Section 3 we study a consequence of Theorem 2, which relates to the type of question studied in [DR10, LHF11]. We give a partial resolution of the problem: given a set $S \subset \mathbb{N}_0^k$, is there an invariant subtraction game $\mathcal{M}$ such that $\mathcal{P}(\mathcal{M}) = S$?

Example 1. *In Figure 1, by Theorem 2 (c), $\mathcal{M}$ is non-reflexive since $(1,2) \ominus (1,1) = (0,1) \in \mathcal{P}(\mathcal{M})$. Neither is the dual, $\mathcal{M}^\star$, since $(1,0)$ and $(3,2)$ are moves, but $(3,2) \ominus (1,0) = (2,2) \in \mathcal{P}(\mathcal{M}^\star)$. On the other hand $\mathcal{M}^{\star\star} = \{(1,1)(2,2)\}$ is reflexive, since $(2,2) \ominus (1,1) = (1,1) \in \mathcal{M}^{\star\star} \subset \mathcal{N}(\mathcal{M}^{\star\star})$. Hence $\mathcal{M}^n$ is reflexive for all $n \geq 2$.*

Example 2. *In Figure 2, notice that $(3,5) \ominus (2,2) = (1,3) \in \mathcal{P}(\mathcal{M})$, so that by Theorem 2 (c), $\mathcal{M}$ is non-reflexive (as is also clear by the figures). However, due to these experimental results, $\mathcal{M}^n \cap \{(i,j) \mid i,j \in \{0,1,\dots,100\}$ is identical for $n = 8$ and $n = 10$ and hence, for all even $n \geq 8$ (and similarly for all odd $n \geq 9$). Of course, by Theorem 1, we get that $\lim \mathcal{M}^{2i}$ exists. However, we do not know whether there exists an $n \geq 8$ such that $\mathcal{M}^n = \lim \mathcal{M}^{2i}$ (see also Question 2 on page 14).*

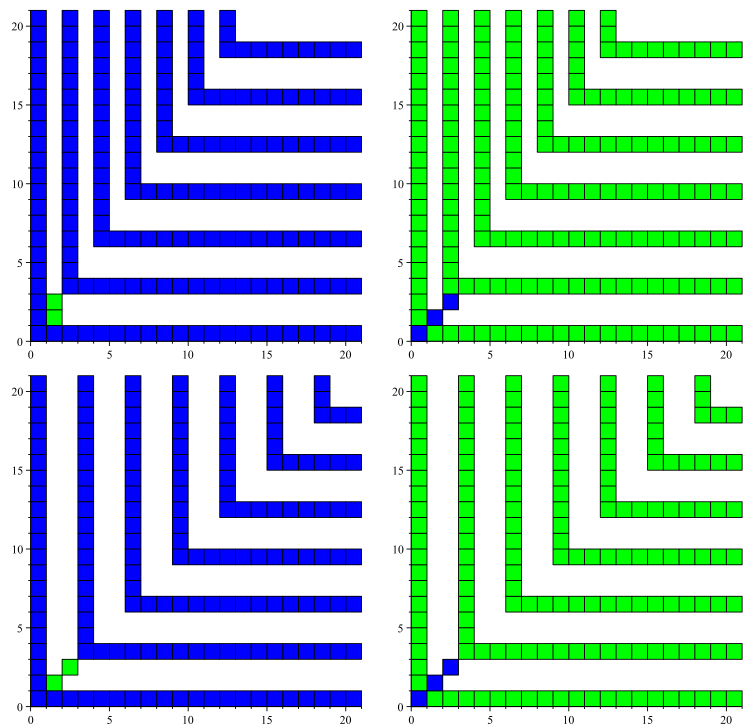

Figure 1: The figures illustrate three recursive applications of the $\star$-operator on $\mathcal{M} = \{(1,1),(1,2)\}$ (for positions with coordinates less than 20). In the upper left figure the green squares represent the two moves in $\mathcal{M}$ and the repetitive blue pattern its (initial) set of P-positions; the upper right figure illustrates the repetitive patterns in $\mathcal{M}^\star$ with its (finite) set of P-positions, and so on.

2 Convergence

Let us begin by proving Theorem 1. The first item in the next lemma is also proved in [LHF11].

Lemma 1 ([LHF11])**.** *Let $\mathcal{M}$ denote an invariant subtraction game. Then*

(a) $\mathcal{P}(\mathcal{M}) \cap \mathcal{M} = \emptyset$*,*

(b) $\mathcal{M}^\star \cap \mathcal{M} = \emptyset$*, and*

(c) $\mathcal{P}(\mathcal{M}) \cap \mathcal{P}(\mathcal{M}^\star) = \{\mathbf{0}\}$*.*

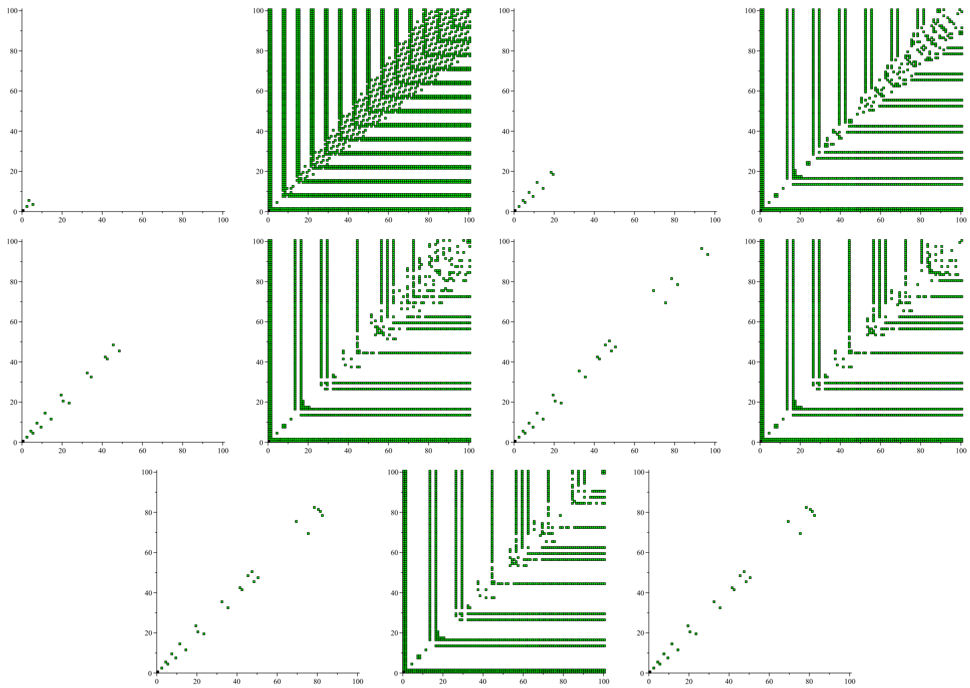

Figure 2: The upper left figure represents the invariant subtraction game $\mathcal{M} = \{(2,2),(3,5),(5,3)\}$. The following figures illustrate 10 recursive applications of the $\star$-operator on this game (for coordinates less than 100).

Proof. Let $\boldsymbol{m} \in \mathcal{M}$ and note that $\boldsymbol{m} \ominus \boldsymbol{m} = \boldsymbol{0} \in \mathcal{P}(\mathcal{M})$, which gives $\boldsymbol{m} \in \mathcal{N}(\mathcal{M})$. This proves (a). By the definition of the $\star$-operator we have that $\mathcal{M}^\star = \mathcal{P}(\mathcal{M})'$. Hence (a) gives (b) and (c). $\square$

The next lemma concerns consequences of Lemma 1 for the $\star\star$-operator.

Lemma 2. *Let $\mathcal{M}$ denote an invariant subtraction game.*

(a) Suppose that $\boldsymbol{x} \in \mathcal{M} \setminus \mathcal{M}^{\star\star}$. Then $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}^\star) \setminus \mathcal{M}^\star$.

(b) Suppose that $\boldsymbol{0} \prec \boldsymbol{x} \in \mathbb{N}_0^k$ is such that, for all $\boldsymbol{m} \prec \boldsymbol{x}$, $\boldsymbol{m} \in \mathcal{M}$ if and only if $\boldsymbol{m} \in \mathcal{M}^{\star\star}$. Then

$$\boldsymbol{x} \notin \mathcal{M}^{\star\star} \setminus \mathcal{M}. \tag{1}$$

Proof. Assume that the hypothesis of item (a) holds. Then, since $\boldsymbol{x} \in \mathcal{M}$, by Lemma 1 (a), $\boldsymbol{x} \notin \mathcal{P}(\mathcal{M})$, so that $\boldsymbol{x} \notin \mathcal{M}^\star$. Also, since $\boldsymbol{x} \notin \mathcal{M}^{\star\star}$, by definition of $\star$, we get that $\boldsymbol{x} \in \mathcal{N}(M^\star)$.

For (b), suppose that the negation of (1) holds, that is that $\boldsymbol{x} \in \mathcal{M}^{\star\star} \setminus \mathcal{M}$. Then

$$\boldsymbol{x} \in \mathcal{P}(\mathcal{M}^\star)', \tag{2}$$

which, by Lemma 1 (c), gives $\boldsymbol{x} \notin \mathcal{P}(\mathcal{M})$. Altogether, we get that $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}) \setminus \mathcal{M}$. Then, by definition of N, there is a move, say $\boldsymbol{m} \in \mathcal{M}$, with $\boldsymbol{m} \prec \boldsymbol{x}$, such that

$$\boldsymbol{y} = \boldsymbol{x} \ominus \boldsymbol{m} \in \mathcal{P}(\mathcal{M})' = \mathcal{M}^\star.$$

By the assumption in the lemma we have that $\boldsymbol{m} \in \mathcal{M}^{\star\star} = \mathcal{P}(\mathcal{M}^\star)'$. Hence, $\boldsymbol{m} = \boldsymbol{x} \ominus \boldsymbol{y}$ is a P-position in $\mathcal{M}^\star$ and, since $\boldsymbol{y} \in \mathcal{M}^\star$, $\boldsymbol{x}$ is an N-position in $\mathcal{M}^\star$, which contradicts (2). □

Proof (of Theorem 1). Let $\mathcal{M}$ denote an invariant subtraction game. Suppose that

$$\boldsymbol{x} \in \mathbb{N}_0^k \setminus \{\boldsymbol{0}\} \tag{3}$$

is such that, for all $\boldsymbol{y} \prec \boldsymbol{x}$,

$$\boldsymbol{y} \in \mathcal{M} \text{ if and only if } \boldsymbol{y} \in \mathcal{M}^{\star\star}. \tag{4}$$

Then clearly

$$\boldsymbol{y} \in \mathcal{P}(\mathcal{M}) \text{ if and only if } \boldsymbol{y} \in \mathcal{P}(\mathcal{M}^{\star\star}), \tag{5}$$

so that, by definition of $\star$,

$$\boldsymbol{y} \in \mathcal{M}^\star \text{ if and only if } \boldsymbol{y} \in \mathcal{M}^3 \tag{6}$$

and hence

$$\boldsymbol{y} \in \mathcal{P}(\mathcal{M}^\star) \text{ if and only if } \boldsymbol{y} \in \mathcal{P}(\mathcal{M}^3). \tag{7}$$

Therefore, a repeated application of $\star$ gives

$$\boldsymbol{y} \in \mathcal{M}^{2i} \text{ if and only if } \boldsymbol{y} \in \mathcal{M}^{2i+2}$$

and also

$$\boldsymbol{y} \in \mathcal{M}^{2i+1} \text{ if and only if } \boldsymbol{y} \in \mathcal{M}^{2i+3},$$

for all $i \in \mathbb{N}_0$.

Suppose that $\boldsymbol{x}$ is of the form in (3) and (4). Then, by the definition of convergence, it suffices to demonstrate that the minimum value $i = i(x)$ for which

$$\boldsymbol{x} \in \mathcal{M}^{2i} \text{ if and only if } \boldsymbol{x} \in \mathcal{M}^{2i+2} \tag{8}$$

is bounded. Precisely, we will show that $i = 1$ suffices, which means that to satisfy (8), at most 2 iterations of $\star\star$ is needed, for each position which satisfies the requirements of $\boldsymbol{x}$ in (4). We then get that, for any game $\mathcal{M}$ and any position $\boldsymbol{x}$, it suffices to take $n_0 = 2\prod_{i=1}^{k} x_i$ in the definition of convergence.

We have four cases,

(A) $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}) \cap \mathcal{N}(\mathcal{M}^{\star\star})$,

(B) $\boldsymbol{x} \in \mathcal{P}(\mathcal{M}) \cap \mathcal{P}(\mathcal{M}^{\star\star})$,

(C) $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}) \cap \mathcal{P}(\mathcal{M}^{\star\star})$ or

(D) $\boldsymbol{x} \in \mathcal{P}(\mathcal{M}) \cap \mathcal{N}(\mathcal{M}^{\star\star})$.

At first, notice that (B) together with Lemma 1 (a) implies $\boldsymbol{x} \notin \mathcal{M} \cup \mathcal{M}^{\star\star}$ (which gives $i = 0$ in (8)). Similarly, for case (D), by using Lemma 1 (a) twice, since $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})' = \mathcal{M}^{\star}$, we get $\boldsymbol{x} \notin \mathcal{M}$ and $\boldsymbol{x} \notin \mathcal{P}(\mathcal{M}^{\star})' = \mathcal{M}^{\star\star}$ (which again gives $i = 0$ in (8)).

It remains to investigate case (A) and (C).

Case (A): By Lemma 2 (b), we have that $\boldsymbol{x} \notin \mathcal{M}^{\star\star} \setminus \mathcal{M}$. Therefore, we may assume that

$$\boldsymbol{x} \in \mathcal{M} \setminus \mathcal{M}^{\star\star} \tag{9}$$

since otherwise we are done. By Lemma 2 (a), this gives that

$$\boldsymbol{x} \in \mathcal{N}(\mathcal{M}^{\star}) \setminus \mathcal{M}^{\star}. \tag{10}$$

Hence, by definition of N in $\mathcal{M}^\star$, we get that there is a position $\boldsymbol{y} \in \mathcal{P}(\mathcal{M}^\star)'$ such that

$$\boldsymbol{m} = \boldsymbol{x} \ominus \boldsymbol{y} \in \mathcal{M}^\star. \tag{11}$$

By (6) this implies that $\boldsymbol{m} \in \mathcal{M}^3$ and by (7) that $\boldsymbol{y} \in \mathcal{P}(\mathcal{M}^3)$. Thus, by definition of P in $\mathcal{M}^3$, the equality in (11) implies that $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}^3)$. Hence, by the definition of the $\star$-operator, we have that $\boldsymbol{x} \notin \mathcal{M}^4$, which, by the assumption (9), suffices for convergence.

Case (C): Since $\boldsymbol{x} \in \mathcal{N}(\mathcal{M})$, the definition of $\star$ gives $\boldsymbol{x} \notin \mathcal{M}^\star$. Hence, by $\boldsymbol{x} \in \mathcal{P}(\mathcal{M}^{\star\star})$ and Lemma 1 (c), since $\boldsymbol{x} \succ \boldsymbol{0}$, we get that $\boldsymbol{x} \notin \mathcal{P}(\mathcal{M}^\star)$ and thus $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}^\star) \setminus \mathcal{M}^\star$. As in the proof of (A), from (10) onwards, this gives that $\boldsymbol{x} \notin \mathcal{M}^4$. Also, Lemma 1 (a), gives that $\boldsymbol{x} \notin \mathcal{M}^{\star\star}$, which proves convergence. □

3 Reflexivity

In this section we discuss criteria for reflexivity of a game. We begin by proving Theorem 2. Let us restate it.

Theorem 2. *Let $\mathcal{M}$ denote an invariant subtraction game. Then the following items are equivalent.*

(a) $\mathcal{M}$ is reflexive,

(b) $\mathcal{M} = \lim_{i\in\mathbb{N}_0} \mathcal{X}^{2i}$, for some invariant subtraction game $\mathcal{X} = \mathcal{X}^0$,

(c) $\mathcal{D}(\mathcal{M}) \subseteq \mathcal{N}(\mathcal{M})$.

Proof. If $\mathcal{M} = \mathcal{M}^{\star\star}$ then $\mathcal{M}^{2i} = \mathcal{M}^{2i+2}$, for all $i \geq 0$, so that $\lim \mathcal{M}^{2i} = \mathcal{M}$. If $\mathcal{M} = \lim \mathcal{M}^{2i}$ exists, then $\mathcal{M}^{\star\star} = (\lim \mathcal{M}^{2i})^{\star\star} = \lim \mathcal{M}^{2i} = \mathcal{M}$. Hence, it remains to prove that $\mathcal{M}$ is reflexive if and only if $D(\mathcal{M}) \subseteq \mathcal{N}(\mathcal{M})$.

"⇒": Suppose that $\mathcal{M}$ is reflexive. Then, we have to prove that $\mathcal{D}(\mathcal{M}) \subseteq \mathcal{N}(\mathcal{M})$. Suppose, on the contrary, that there are distinct $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{M}$ such that

$$\boldsymbol{m}_1 \ominus \boldsymbol{m}_2 = \boldsymbol{x} \in \mathcal{P}(\mathcal{M})'. \tag{12}$$

Then, by definition of $\star$,

$$\boldsymbol{x} \in \mathcal{M}^\star. \tag{13}$$

Also, by reflexivity, we get that $\{\boldsymbol{m}_1, \boldsymbol{m}_2\} \subset \mathcal{M}^{\star\star} = \mathcal{P}(\mathcal{M}^\star)'$. But, by (12) and (13), this means that there is a move from a P-position to another P-position in $\mathcal{M}^\star$, which is impossible.

"$\Leftarrow$": Suppose that $\mathcal{D}(\mathcal{M}) \subseteq \mathcal{N}(\mathcal{M})$ but $\mathcal{M} \neq \mathcal{M}^{\star\star}$. Then there is some least $\boldsymbol{m} \in (\mathcal{M} \setminus \mathcal{M}^{\star\star}) \cup (\mathcal{M}^{\star\star} \setminus \mathcal{M})$, which, by Lemma 2 (b), gives $\boldsymbol{m} \in \mathcal{M} \setminus \mathcal{M}^{\star\star}$. By Lemma 2 (a), we get $\boldsymbol{m} \in \mathcal{N}(\mathcal{M}^\star) \setminus \mathcal{M}^\star$. Then, by definition of N in $\mathcal{M}^\star$, there is an $\boldsymbol{x} \in \mathcal{M}^\star$ such that

$$\boldsymbol{m} \ominus \boldsymbol{x} = \boldsymbol{y} \in \mathcal{P}(\mathcal{M}^\star)'. \tag{14}$$

Then, by definition of $\star$, we get $\boldsymbol{y} \in \mathcal{M}^{\star\star}$ and so, by minimality of $\boldsymbol{m}$, $\boldsymbol{y} \in \mathcal{M} \cap \mathcal{M}^{\star\star}$, so that both $\boldsymbol{m}$ and $\boldsymbol{y}$ are moves in $\mathcal{M}$. But then (14) together with the definition of $\boldsymbol{x}$ and the $\star$-operator give $\boldsymbol{m} \ominus \boldsymbol{y} = \boldsymbol{x} \in \mathcal{P}(\mathcal{M})$, which contradicts $\mathcal{D}(\mathcal{M}) \subseteq \mathcal{N}(\mathcal{M})$. □

By Theorem 2 (c), one never needs to compute $\mathcal{P}(\mathcal{M}^\star)$ to decide whether $\mathcal{M}$ is reflexive or not. Sometimes a very incomplete understanding of the winning strategy $\mathcal{P}(\mathcal{M})$ suffices. Namely, to disprove reflexivity of $\mathcal{M}$ it suffices to find a single P-position $\boldsymbol{x} \succ \boldsymbol{0}$ which connects any two moves $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{M}$ in the sense that $\boldsymbol{x} = \boldsymbol{m}_1 \ominus \boldsymbol{m}_2$. If $\mathcal{M}$ were reflexive this would imply $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{M}^{\star\star} = \mathcal{P}(\mathcal{M}^\star)'$, with $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})' = \mathcal{M}^\star$, which is impossible. See also Example 4. On the other hand, to prove reflexivity, it suffices to find some subset $X \subseteq \mathcal{N}(\mathcal{M})$ such that $\mathcal{D}(\mathcal{M}) \subseteq X$ holds.

In particular, if we can take $X = \mathcal{M}$ we obtain very simple reflexivity properties. Namely, whenever $\mathcal{D}(\mathcal{M}) \subseteq \mathcal{M}$, the game $\mathcal{M}$ is 'trivially' reflexive, that is, for this case we do not even need to study $\mathcal{P}(\mathcal{M})$ to establish reflexivity.

Let $X \subseteq \mathbb{N}_0^k$. Then the set X is

- *subtractive* if, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, with $\boldsymbol{x} \prec \boldsymbol{y}$, $\boldsymbol{y} \ominus \boldsymbol{x} \in X$.
- a *lower ideal* if, for all $\boldsymbol{y} \in X$, $\boldsymbol{x} \prec \boldsymbol{y}$ implies $\boldsymbol{x} \in X$. (Hence the set of terminal P-positions of a given invariant subtraction game constitutes a lower ideal.)

- an *anti-chain*, if all distinct pairs $\boldsymbol{x}, \boldsymbol{y} \in X$ are unrelated, that is $\boldsymbol{x} \preceq \boldsymbol{y}$ implies $\boldsymbol{x} = \boldsymbol{y}$.

We have the following corollary of Theorem 2 (see also Figure 3 for an application of (a)).

Corollary 1. *The invariant subtraction game $\mathcal{M}$ is reflexive if, regarded as a set,*

(a) $\mathcal{M}$ *is subtractive,*

(b) $\mathcal{M}$ *is a lower ideal,*

(c) $\mathcal{M} = \{(x, 0, \ldots, 0), (0, x, 0, \ldots, 0), \ldots, (0, \ldots, 0, x) \in \mathbb{N}_0^k \mid x \in \mathbb{N}\}$*, that is* $\mathcal{M}$ *represents the classical game of* k*-pile Nim [B1902],*

(d) $\mathcal{M}$ *is an anti-chain, or*

(e) $\mathcal{M} \in \{\emptyset, \{\boldsymbol{m}\}\}$*, that is* $\mathcal{M}$ *consists of at most a single move.*

Proof. For (a), notice that

$$\mathcal{D}(\mathcal{M}) = \{\boldsymbol{m}_1 \ominus \boldsymbol{m}_2 \succ \boldsymbol{0} \mid \boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{M}\} \subseteq \mathcal{M} \subseteq \mathcal{N}(\mathcal{M}),$$

which, by Theorem 2, gives the claim. Then, the inclusions of families of games $\{\mathcal{M}_e\} \subseteq \{\mathcal{M}_d\} \subseteq \{\mathcal{M}_a\}$ and $\{\mathcal{M}_c\} \subseteq \{\mathcal{M}_b\} \subseteq \{\mathcal{M}_a\}$ prove the corollary, where $\mathcal{M}_i$ denotes the game given by the set $\mathcal{M}$ as in item (i). $\square$

Example 3. *In Figure 1,* $\mathcal{M}^{\star\star} = \{(1,1), (2,2)\}$ *is subtractive and hence, by Corollary 1, reflexive, but* $\mathcal{M} = \{(1,1), (1,2)\}$ *is neither. For another example, the invariant subtraction game* $\mathcal{M} = \{(1,1), (2,2), (0,8), (8,0)\}$ *is subtractive and hence reflexive. Hence its dual game* $\mathcal{M}^{\star} = \mathcal{P}(\mathcal{M})'$ *is also reflexive (but not subtractive). Figure 3 represents the first few moves of* $\mathcal{M}^{\star} = \{(1,1), (2,2), (0,8), (8,0)\}^{\star}$*. In spite of the simplicity of the game* $\mathcal{M}$*, the P-positions seem to have a very complex structure (in the sense of [F04]). It seems to be a-periodic in general, but asymptotically periodic for each fixed* x*-coordinate (or* y*-coordinate), but we do not understand these patterns yet. See also the final section for a comment regarding undecidability of games with a finite number of moves.*

We believe that there are many more interesting applications of Theorem 2. Let us begin with two of them.

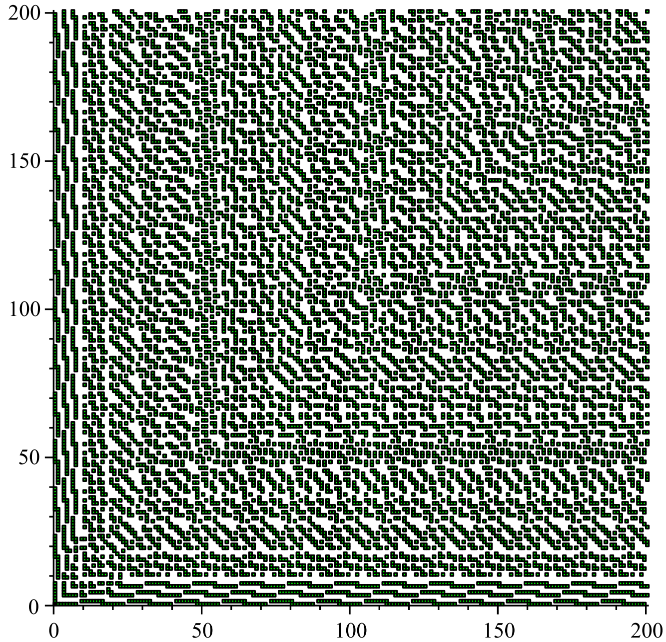

Figure 3: The dual game $\mathcal{M}^\star$ for the invariant subtraction game $\mathcal{M} = \{(1,1),(2,2),(0,8),(8,0)\}$.

3.1 A consequence of reflexivity

Given a 'candidate' set $\mathbf{0} \in S \subset \mathbb{N}_0^k$ of P-positions, is there an invariant subtraction game $\mathcal{M}$ such that $\mathcal{P}(\mathcal{M}) = S$? This type of question was introduced in [DR10], together with a challenging conjecture on a family of sets $S \subset \mathbb{N}_0^2$ defined by a certain class of increasing sequences of positive integers. (The conjecture was resolved in [LHF11].) As a consequence of Theorem 2 (and Corollary 1), we are able to shed some new light on this type of question for general sets S.

Corollary 2. *Let $\mathbf{0} \in S \subset \mathbb{N}_0^k$, $k \in \mathbb{N}$. If the invariant subtraction game S' is reflexive, so that, by Theorem 2,*

$$\mathcal{D}(S) \subseteq \mathcal{N}(S'), \tag{15}$$

then there is an invariant subtraction game $\mathcal{M}$ satisfying

$$\mathcal{P}(\mathcal{M}) = S. \tag{16}$$

Specifically, one such game $\mathcal{M}$ is given by the recursive construction which defines the set of P-positions of the invariant subtraction game S'.

Proof. Suppose that (15) holds and take $\mathcal{M} = \mathcal{P}(S')' = (S')^\star$. Then, since $S' = (S')^{\star\star}$, $\mathcal{P}(\mathcal{M})' = \mathcal{P}((S')^\star)' = (S')^{\star\star} = S'$ gives the claim. □

It is easy to find sets S which do not satisfy (16) for any $\mathcal{M}$ (and where the invariant subtraction game S' is non-reflexive). See also [DR10, LHF11] and [G66, Theorem 3.2] for related results.

Example 4. *Let $S' = \{(1,1),(1,2)\}$ (see also Example 1 and Figure 1). Then $\mathcal{D}(S') = \{(0,1)\} \subset \{(0,x) \mid x \in \mathbb{N}_0\} \subset \mathcal{P}(S')$ so that reflexivity of S' does not hold. Further, for this choice of S, there is no invariant subtraction game $\mathcal{M}$ which satisfies (16). Indeed, by the definition of N, since $(0,1)$ is not a (candidate) P-position, it has to be a move in $\mathcal{M}$. But this contradicts the definition of P since $(1,2) \ominus (1,1) = (0,1)$.*

On the other hand, Figure 1 also illustrates that a non-reflexive game, namely $\mathcal{M}^\star$, might produce a reflexive $S' = \mathcal{M}^{\star\star}$ (Wythoff Nim is another such example [LHF11]), see also Question 2. However it is not necessary that S' is reflexive for (16) to hold. A non-reflexive $\mathcal{M}$ can produce a non-reflexive S' as we have seen in Figure 1 (take $S' = \mathcal{M}^\star$) and also in Figure 2 (take $S' = \mathcal{M}^i$, many i).

Let us give another example of a non-reflexive game S' which satisfies (16). We believe that strictly more than two P-positions are needed for such examples to hold.

Example 5. *Suppose that $S' = \{(0,1),(1,0),(1,1),(3,3)\}$. Then Corollary 1 does not give any information on whether there is an invariant subtraction game $\mathcal{M}$ such that (16) holds. Namely we have that $(2,2) \in \mathcal{D}(S) \cap \mathcal{P}(S')$, which contradicts (15) (and thus reflexivity of S'). However, by inspection one finds that $S \subset \mathcal{P}(\mathcal{Q})$ for $\mathcal{Q} = \{(0,2),(2,0),(1,2),(2,1)\}$. Then, in spite of the observation that S' is non-reflexive, this gives the existence of a game $\mathcal{M}$ satisfying (16). (For example take $\mathcal{M} = \mathcal{Q} \cup \{(x,y),(y,x) \mid x \geq 4\}$.)*

3.2 Decidability and reflexivity

A very simple configuration of moves, e.g. as in Figure 3, can have a very complex set of P-positions (dual game). In fact, suppose the invariant subtraction game $\mathcal{M} \subset \mathbb{N}_0^k$ has finite cardinality. Then we wonder whether it is algorithmically decidable if a given k-tuple ($\succ \mathbf{0}$) appears as a difference of any two P-positions in $\mathcal{M}$; that is if the set of P-position changes if we 'modify' an invariant subtraction game $\mathcal{M}$ and rather play $\mathcal{M} \cup \{\boldsymbol{m}\}$, $\boldsymbol{m} \in \mathbb{N}_0^k$. (In [LW] we prove undecidability in a related sense for a similar class of invariant games.)

However, by Theorem 2, since $\mathcal{D}(\mathcal{M})$ is finite whenever $\mathcal{M}$ is, it takes at most a finite computation to decide whether $\mathcal{M}$ is reflexive or not. Hence we get another corollary of Theorem 2.

Corollary 3. *Suppose that the number of moves in the invariant subtraction game $\mathcal{M}$ is finite. Then the problem of determining whether the game $\mathcal{M}$ is reflexive or not is algorithmically decidable.*

4 Discussion

In this paper we have presented some general territory of invariant subtraction games and the $\star$-operator. The issues of convergence of the $\star\star$-operator have been completely resolved, but we have not found any explicit formula for a 'non-trivial limit-game'. By 'trivial limit-game' we here mean a game H which satisfies $H = \mathcal{M}^{2n} = \lim \mathcal{M}^{2i}$ for some $n \in \mathbb{N}$ and some game $\mathcal{M}$.

Problem 1. *Prove or disprove that all limit games are trivial. In the latter case give an explicit formula for a non-trivial limit game without the mention of a limit of a sequence of games.*

Our next question is a continuation of the examples in Section 3.

Question 1. *Examples 4 and 5 suggest a classification of 'non-reflexive' sets $S' \subset \mathbb{N}_0^k$, that is, by Theorem 2, sets for which there exists a pair $\boldsymbol{x}, \boldsymbol{y} \in S'$ such that $\boldsymbol{x} \ominus \boldsymbol{y} \in \mathcal{P}(S')'$. The first class should contain those sets S for which there exist an invariant subtraction game $\mathcal{M}$ such that $\mathcal{P}(\mathcal{M}) = S$ and the second, those for which there is no such game. Suppose there exists a pair $\boldsymbol{x}, \boldsymbol{y} \in S'$ such that the only possible 'candidate move' from $\boldsymbol{m} = \boldsymbol{x} \ominus \boldsymbol{y}$ to another position in S is to $\mathbf{0}$. Then, we are in Example 4 and so in the second class. On the other hand, Example 5 gives an example when there is no such pair $\boldsymbol{x}, \boldsymbol{y}$. But suppose that the positions $(2,3)$ and $(3,2)$ are included to the set S in Example 5. Then, neither the move $(2,2)$ nor the moves $(1,2)$ and $(2,1)$ may be included to the candidate set $\mathcal{M}$, and hence S would have belonged to the second class. Is there an explicit and exhaustive classification which settles the type of question suggested by Example 4 and 5?*

In Figure 1 we gave an example of a non-reflexive game with a non-reflexive dual, but where the dual of the dual is reflexive. The example of the 'symmetric' game $\mathcal{M} = \{(2,2),(3,5),(5,3)\}$ from Figure 2 contains only three moves, but we were not able to determine whether there is an n such that $\mathcal{M}^n$ is reflexive or not. This discussion leads us to our final question.

Question 2. *Is there, for each $n \in \mathbb{N}$, a game $\mathcal{M}$ such that $\mathcal{M}^n$ is reflexive, but $\mathcal{M}^{n-1}$ is not?*

We do not know if the answer to Question 2 is positive for any $n \geq 3$.

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