Maharaja Nim

Wythoff's Queen meets the Knight

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Abstract

We relax the hypothesis of a recent result of A. S. Fraenkel and U. Peled on certain complementary sequences of positive integers. The motivation is to understand to asymptotic behavior of the impartial game of *Maharaja Nim*, an extension of the classical game of Wythoff Nim. In the latter game, two players take turn in moving a single Queen of Chess on a large board, attempting to be the first to put her in the lower left corner, position $(0, 0)$. Here, in addition to the classical rules, a player may also move the Queen as the Knight of Chess moves, still taking into consideration that, by moving no coordinate increases. We prove that the second player's winning positions are close to those of Wythoff Nim, namely they are within a bounded distance to the half-lines, starting at the origin, of slope $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}-1}{2}$ respectively. We encode the patterns of the P-positions by means of a certain *dictionary process*, thus introducing a new method for analyzing games related to Wythoff Nim. Via Post's Tag productions, we also prove that, in general, such dictionary processes are algorithmically undecidable. Keywords: Approximate linearity, complementary sequences, dictionary process, impartial game, Wythoff Nim, game complexity.

1 Maharaja Nim

We introduce a 2-player combinatorial game called *Maharaja Nim*, an extension of the well-known game of Wythoff Nim [Wy1907]. (The name Maharaja is taken from a variation of Chess, "The Maharaja and the Sepoys", [Fa].) Both these games are impartial, that is, the set of move options are the same regardless of whose turn it is. For a background on impartial games see [BCG].

We let $\mathbb{N}$ and $\mathbb{N}_0$ denote the positive and nonnegative integers respectively. It is convenient to label our positions by ordered pairs of nonnegative integers. Place a *Queen of Chess* on a given position (x, y), $x, y \in \mathbb{N}_0$, of a large (Chess) board, with the position in the lower left corner the unique terminal position, labeled $(0, 0)$. In the game of Wythoff Nim, here denoted by W, the two players move the Queen alternately as it moves in Chess, but with the restriction that, by moving, no coordinate increases, see Figure 1. A player who cannot move, because the position is $(0, 0)$, loses. (This variation of Wythoff Nim is called "Corner the Queen" and was invented by R. P. Isaacs in 1960.) In Maharaja Nim, denoted by M, the rules are as in Wythoff Nim, except that the Queen is exchanged for a *Maharaja*, a piece which may move both as the Queen and the *Knight* of Chess, again, provided by moving no coordinate increases, see Figure 1. Hence, for $x, y \in \mathbb{N}_0$, we get that, if (x, y) is a given position of Wythoff Nim, then its options are of the forms $(x, y - r), (x - s, y)$ and $(x - t, y - t)$, for $0 < r \leq y$, $0 < s \leq x$ and $0 < t \leq \min\{x, y\}$ respectively. For Maharaja Nim the two options $(x - 1, y - 2)$ and $(x - 2, y - 1)$ are also available, provided the respective coordinates are nonnegative.

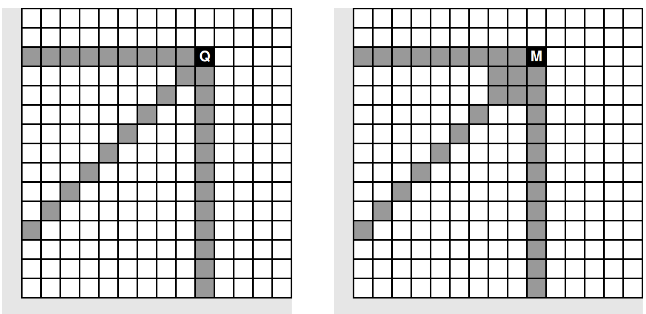

Figure 1: The move options, from a given position, of Wythoff Nim and Maharaja Nim respectively.

As usual, for impartial games, we denote a position by P if the second player wins, otherwise N. Our games will terminate in a finite number of moves so that the sets of P- and N-positions will partition the set of starting positions. We let $\mathcal{P}_{\mathrm{M}}$ and $\mathcal{P}_{\mathrm{W}}$ denote the set of P-positions of Maharaja Nim and Wythoff Nim respectively. See Figure 2 for a computation of the initial P-positions of the respective games and the Appendix, Section A, for the corresponding code.

Let $\phi = \frac{1+\sqrt{5}}{2}$ denote the *Golden ratio*. Wythoff Nim's set pf P-positions is

$$\mathcal{P}_{\mathrm{W}} = \{(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor), (\lfloor \phi^2 n \rfloor, \lfloor \phi n \rfloor) \mid n \in \mathbb{N}_0\}, \tag{1}$$

[Wy1907]. From this it follows that there is precisely one P-position of Wythoff Nim in each row and each column of the board (see also [Be1926]).

The purpose of this paper is to explore the P-positions of Maharaja Nim. In particular we are interested in their relation to the Golden ratio. In a sense, we will prove that the (asymptotic) behavior of the P-positions of Wythoff Nim remains stable when the Knight type moves are adjoined to those of the Queen, namely they will remain 'close' to the half-lines, starting at the origin, of slope ϕ^{-1} and ϕ respectively (see Figure 3). We let $O(1)$ denote bounded functions on $\mathbb{N}_0$.

Theorem 1.1 (Main Theorem)**.** *Each P-position of Maharaja Nim lies on one of the stripes $\phi x + O(1)$ or $\phi^{-1}x + O(1)$, that is, if $(x, y) \in \mathcal{P}_M$, with $y \geq x$, then $y - \phi x$ is $O(1)$.*

We prove this result in Section 3, by encoding the patterns of the P-postions by means of a certain *dictionary process*, thus introducing a new method for analyzing games related to Wythoff Nim. This result is preceded by some general properties of our games as well as a number theoretical Central Lemma in Section 2. In Section 4 we finish off by proving that our dictionary processes are in general *algorithmically undecidable.*

2 Complementary sequences and a central lemma

Let us begin by discussing some of the main properties of the P-positions of our games. Clearly $(0, 0)$ is P. Another trivial observation is that, since the rules of game are symmetric, if (x, y) is P then (y, x) is P. It is also easy to see that

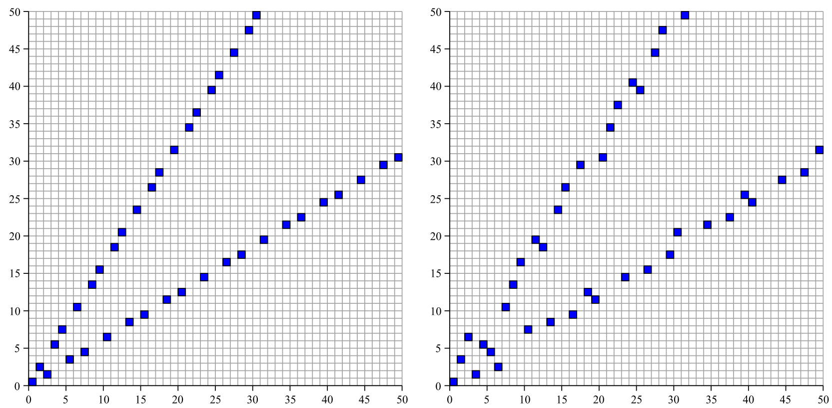

Figure 2: The initial P-positions of Wythoff Nim and Maharaja Nim respectively.

there is at most one P-position in each row and each column (corresponding to the Rook-type moves). But, in fact, the same assertion as for Wythoff Nim holds:

Proposition 2.1. *There is precisely one P-position of Maharaja Nim in each row and each column of* $\mathbb{N}_0 \times \mathbb{N}_0$.

Proof. Since all Nim-type moves are allowed in Maharaja Nim, there is at most one P-position in each row and column of $\mathbb{N}_0 \times \mathbb{N}_0$. This implies that there are at most k P-positions strictly to the left of the kth column (row). Each such P-position is an option for at most three N-positions in column (row) k. This implies that there is a least position in column (row) k which has only N-positions as options. By definition this position is P and so, since k is an arbitrary index, the result follows. ■

Another claim holds for both Wythoff Nim and Maharaja Nim. There is at most one P-position on each (upper) *diagonal* of the form

$$\{(x, x + C) \mid x \in \mathbb{N}_0\}, C \in \mathbb{N}_0, \tag{2}$$

(corresponding to the Bishop-type moves). We call (2) the *Cth diagonal*. By symmetry it suffices to consider the upper diagonals.

For Wythoff Nim, (1) readily gives that there is *precisely* one P-position on

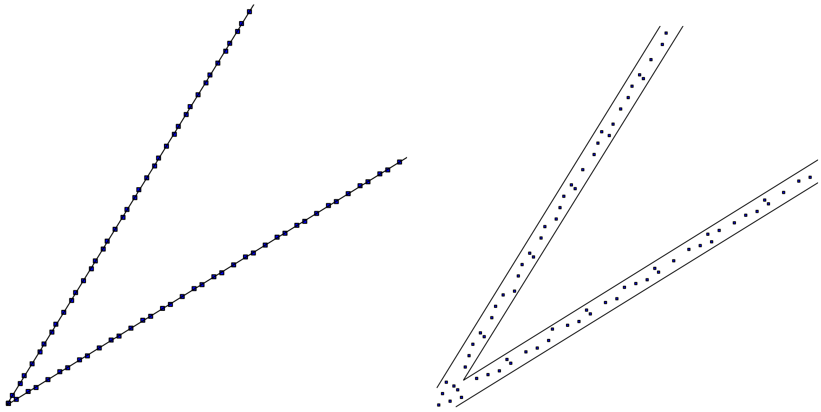

Figure 3: To the left, the P-positions of Wythoff Nim lie 'on' the half-lines ϕx and $\phi^{-1}x$, $x \geq 0$. The figure to the right illustrates a main result of this paper, that the P-positions of Maharaja Nim are bounded below and above by the stripes $y = \phi x + O(1)$ and $y = \phi^{-1}x + O(1)$ respectively.

each such diagonal and more is true: if

$$\mathcal{P}_{\mathrm{W}} = \{(a_i, b_i), (b_i, a_i)\}, \tag{3}$$

with (a_i) increasing and for all i, $a_i \leq b_i$, then for all n,

$$\{0, 1, \ldots, n\} = \{b_i - a_i \mid i \in \{0, 1, \ldots, n\}\}. \tag{4}$$

As we will see later in this section, a somewhat weaker, but crucial, property holds also for Maharaja Nim.

We say that two sequences of positive integers are *complementary* if each positive integer is contained precisely once in precisely one of these sequences. In [FP] the authors proves the following result.

Proposition 2.2 (Fraenkel, Peled)**.** *Suppose (x_n) and (y_n) are complementary and increasing sequences of positive integers. Suppose further that there is a positive real constant, δ, such that, for all n,*

$$y_n - x_n = \delta n + O(1). \tag{5}$$

Then there are constants, $1 < \alpha < 2 < \beta$, such that, for all n,

$$x_n - \alpha n = O(1) \tag{6}$$

and

$$y_n - \beta n = O(1). \tag{7}$$

Since, as we will see, the y-sequence of Maharaja Nim's P-positions is not increasing, we cannot use this proposition directly. However, we have found a simplified proof of an extension of this result.

By simple density estimates one may decide the constants α and β, in Proposition 2.2, as functions of δ. Namely, notice that (5) and (6) together imply

$$\beta = \alpha + \delta \tag{8}$$

and, by complementarity, we must have

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1. \tag{9}$$

(Thus α and β are algebraic numbers if and only if δ is.) By this we get the relation

$$\delta(1 - \alpha) + \alpha = (\alpha - 1)\alpha, \tag{10}$$

which will turn out useful. If we denote

$$\mathcal{P}_{\mathrm{M}} = \{(a_n, b_n), (b_n, a_n) \mid n \in \mathbb{N}_0\}, \tag{11}$$

with (a_n) increasing and for all $n, b_n \geq a_n$, then, for all n, b_n is uniquely defined by the rules of M. At this point, one might want to observe that, if the b-sequence would have been increasing (by Figure 2 it is not) then Theorem 1.1 would follow from Proposition 2.2 if one could only establish the following claim: $b_n - a_n - n$ is $O(1)$. Namely in (10) $\delta = 1$ gives $\alpha = \phi$ in Proposition 2.2. Now, interestingly enough, it turns out that Proposition 2.2 holds without the condition that the y-sequence be increasing, namely (5) together with an increasing x-sequence suffices.

Lemma 2.3 (Central Lemma)**.** *Suppose (x_n) and (y_n) are complementary sequences of positive integers with (x_n) increasing. Suppose further that there is a positive real constant, δ, such that, for all n,*

$$y_n - x_n = \delta n + O(1). \tag{12}$$

Then there are constants, $1 < \alpha < 2 < \beta$, such that, for all n,

$$x_n - \alpha n = O(1) \tag{13}$$

and

$$y_n - \beta n = O(1). \tag{14}$$

Proof. We begin by demonstrating that, for all $n \in \mathbb{N}$,

$$x_{n+1} = x_n + O(1), \tag{15}$$

and

$$y_{n+1} = y_n + O(1). \tag{16}$$

By (12), for all $k, n \in \mathbb{N}$ we have that

$$\begin{aligned} y_{n+k} - y_n &= x_{n+k} + \delta(n+k) - x_n - \delta n + O(1), \\ &= x_{n+k} - x_n + \delta k + O(1). \end{aligned} \tag{17}$$

Since for all $k, n \in \mathbb{N}$, $x_{n+k} - x_n \geq k$ and $\delta > 0$ this means that, for all $k, n \in \mathbb{N}$,

$$y_{n+k} \geq y_n - C, \tag{18}$$

where C is some universal positive constant (which may depend on δ). But, with C as in (18), we can find another universal constant $\kappa = \kappa(C) \in \mathbb{N}$ such that, for all n,

$$y_{n+\kappa} - y_n \geq \kappa + 2C + 1. \tag{19}$$

This follows since, in (17), for any C, we can find $k = k(C)$ such that, for all n, $\delta k + O(1) > 2C$. Any such k suffices as our κ. On the one hand there can be at most $\kappa - 1$ numbers from the y-sequence strictly between y_n and $y_{n+\kappa}$ (with indexes strictly in-between n and $n + \kappa$). On the other hand the inequality (18) gives that there can be at most C numbers from the y-sequence with index greater than $n + \kappa$ but less than $y_{n+\kappa}$. It also gives that there can be at most C numbers with index less than n but greater than y_n. Therefore, by complementarity and (19), there has to be a number from the x-sequence in every interval of length $\kappa + 2C + 1$. Thus the jumps in the x-sequence are bounded, which is (15). But then (16) follows from (12) and (15) since

$$\begin{aligned} y_{n+1} - y_n &= x_{n+1} + \delta(n+1) - x_n - \delta n + O(1) \\ &= x_{n+1} + \delta - x_n + O(1) \\ &= O(1). \end{aligned}$$

By (16) we may define m as a function of n with

$$x_n = y_m + O(1). \tag{20}$$

(For example, one can take $m = m(n)$ the least number such that $x_n < y_m$. Then $y_m - x_n$ has to be bounded for otherwise $y_m - y_{m-1}$ is not bounded.) This has two consequences, of which the first one is

$$x_n = n + m + O(1). \tag{21}$$

This follows since the numbers $1, 2, \ldots, x_n$ are partitioned in n numbers from the x-sequence, and the rest, by complementarity, $m + O(1)$ numbers from the y-sequence.

The second consequence of (20) is that, by using (12),

$$x_n = x_m + \delta m + O(1). \tag{22}$$

If $\lim x_n/n$ and $\lim y_n/n$ exist then, clearly they must satisfy (8) and (9) with δ as in the lemma. Thus, using this definition of $\alpha = \alpha(\delta)$, for all n, denote

$$\Delta_n := x_n - \alpha n.$$

We want to use (21) and (22) to express Δ_n in terms of Δ_m.

Equation (22) expresses x_n in terms of x_m and m. Therefore, we wish to combine (21) and (22) to express n in terms of x_m and m, that is, we wish to eliminate x_n from (21). If we plug in the expression (22) for x_n in (21) and solve for n we get

$$n = x_m + (\delta - 1)m + O(1). \tag{23}$$

Combining (22) and (23) gives

$$\begin{aligned} \Delta_n &= x_m + \delta m - \alpha(x_m + (\delta - 1)m) + O(1) \\ &= (1-\alpha)x_m + (\delta(1-\alpha) + \alpha)m + O(1) \\ &= (1-\alpha)\Delta_m + O(1), \end{aligned} \tag{24}$$

where the last equality is by (10).

Notice that, by (22), for sufficiently large n we have that $m < n$. Hence we may use strong induction and by (24) conclude that Δ_n is $O(1)$ which is (13). Then (14) follows from (12). ■

3 Perfect sectors, a dictionary and the proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We begin by proving that there is precisely one P-position of Maharaja Nim on each diagonal of the form in (2). Then we explain how the proof of this result leads to the second part of the theorem, the bounding of the P-positions within the desired *stripes* (Figure 3).

A position, say (x, y), is an *upper* position if it is strictly above the *main diagonal*, that is if $y > x$. Otherwise it is *lower*.

We call a (C, X)-*perfect sector*, or simply a *perfect sector*, all positions strictly above the Cth diagonal, of the form in (2), and strictly to the right of column X. See Figures 4 and 5.

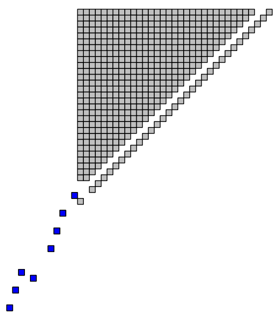

Figure 4: All upper positions from which a player can move to an upper P-position are erased. (The sector continues above the figure.) However, the sector is not perfect.

Suppose that we have computed all P-positions in the columns $1, 2, \ldots, a_{n-1}$ and that, when we erase each upper position from which a player can move to an upper P-position, then the remaining upper positions strictly to the right of

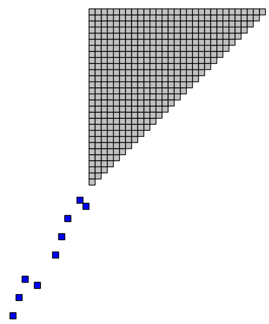

Figure 5: (Step 1) A perfect sector together with the corresponding initial P-positions.

column a_{n-1} constitute an $(n-1, a_{n-1})$-perfect sector (Figure 5). Then we say that a_{n-1} is *perfect* and, in fact, it is easy to see that also property (4) holds for all such n. As we will see, it is essential to our approach that, for any such n,

$$b_n - a_n = n \tag{25}$$

(whenever lower P-positions do not interfere).

Lemma 3.1. *Let $n \in \mathbb{N}$ be sufficiently large so that Knight type moves from lower P-positions do not affect the coordinates of upper P-positions and define (a_i) and (b_i) as in (11). Suppose that a_{n-1} is perfect. Then*

$$\{0, 1, \ldots, n-1\} = \{b_i - a_i \mid 0 \le i < n\} \tag{26}$$

and (25) holds.

Proof. There are precisely $n-1$ upper P-positions. By the Bishop type moves they produce precisely $n-1$ upper diagonals of N-positions between the perfect sector and the 0th diagonal. Hence (26) holds. Further, there is no upper P-position to the left of the a_nth column that interferes with the perfect sector via a Knight type move, because then the sector would not have been perfect. Hence, by definition of P, since the nth upper diagonal is free, the position $(a_n, a_n + n) \in \mathcal{P}_M$. This gives $b_n = a_n + n$ so that (25) holds. ■

We will adjoin a new *word* to Maharaja Nim's *dictionary* if and only if the conditions in this lemma are satisfied. Let us proceed to explain this construction.

3.1 Constructing Maharaja Nim's bit-string

We study a bit-string, a sequence of '0's and '1's, where the ith bit equals '0' if and only if there is an upper P-position of Maharaja Nim in column i. By Proposition 2.1, if there is no upper P-position in column i, there is a lower ditto (the ith bit equals 1).

Suppose (as an induction hypothesis) that column $n-1$ is perfect. Then, by symmetry, we know some lower P-positions in columns to the right of n. The next step is to erase each column in the perfect sector for which there currently is a lower P-position, a '1' in the bit-string (see Figure 6).

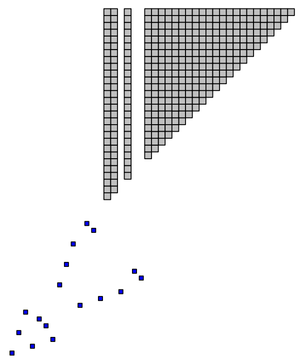

Figure 6: (Step 2) Each column in the perfect sector which corresponds to a lower P-position (a '1' in the bit-string) has been erased.

Then, recursively in the non-erased part of the perfect sector, we compute new upper P-positions until we reach the next perfect sector (for the moment assume that this will happen) at say the perfect column $n-1+m$, $m>0$. Thus, using this notation, we define a *word*, say w, of length m, containing the information of whether the P-position in column $i \in \{n, n+1, \ldots, n+m-1\}$ is below or above the main diagonal.

At this point we adjoin this word together with its unique *translate*, $D(w)$, to Maharaja Nim's *dictionary*. The translate is obtained accordingly: for each P-position in the columns n to $n+m-1$ define the ith bit in $D(w)$ as a '1' if and only if row $k+i$ has an upper P-position and where k is the largest row index strictly below the perfect sector. See also Figure 7 where $k = 12$ in the leftmost picture. The translate $D(w)$ will have length $m + l$, where l denotes the number of '0's in the word w. This follows since, by counting diagonals, a new perfect sector will start in the position $(n + m, k + m + l + 1)$. Since the lth P-position (in case $l > 0$) will not correspond to the upper most P-position, $D(w)$ will end with at least 2 '0's.

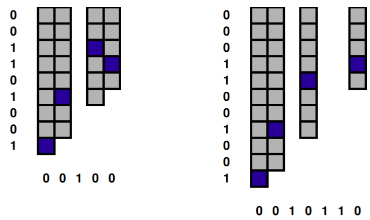

Figure 7: To the left, the unique (upper) P-positions of Maharaja Nim in the columns 8 to 12 are computed. The corresponding translation is 00100 $\rightarrow$ 100101100. To the right are the P-positions in the columns 14 to 20 together with the translation 0010110 $\rightarrow$ 10010011000. (Here we have omitted column 13 with its translation 1 $\rightarrow$ 0.) See also Figure 2 and Section 3.2.

We then concatenate the translate, $D(w)$, at the end of the existing bit-string. In this way, provided a next perfect sector will be detected, the bit-string will always grow faster than we read from it. However, there is no immediate guarantee that we will be able to repeat the procedure—that the next word exists—or for that matter that the size of the dictionary will be finite, so that the process may be described by a finite system of words and translates. But, in the coming, we aim to prove that, in fact, the next perfect sector will always (in the sense outlined above) be detected within a 'period' of at most 7 upper P-positions, that is '0's in the bit-string. As we will see, a complete dictionary needs only (between 9 and) 14 translations. See also Section 4 for a brief general discussion of such dictionary processes. (In this context it is interesting to observe that $\mathcal{P}_W$, see Figure 2, can be derived from the dictionary $0 \rightarrow 10$ and $1 \rightarrow 0$; starting at the first column the word begins 01001010 . . . and hence we get the well known infinite *Fibonacci morphism*.) Let us describe how Maharaja's bit-string is constructed.

3.2 The first translations

Initially, there is some interference which does not allow a recursive definition of words and translates, see Figure 2. The first perfect sector beyond the origin is attained when the 4 first P-positions strictly above the main diagonal has been computed. This happens in column 8, which is Step 1 above. There is only one P-position below the main diagonal, as in Step 2, corresponding to a 1 in the current bit-string. It will erase column 10 in the perfect sector. Then to the right of column 12 a new perfect sector will be detected. Thus the first word (left hand side entry) in the dictionary will be $w = 00100$ (the left picture in Figure 7), corresponding to the P-positions $(8, 13)$, $(9, 16)$, $(10, 7)$, $(11, 19)$ and $(12, 18)$.

Let us verify that this word translates to $D(w) = 100101100$. Notice that the first '1'-bit in $D(w)$ means that the P-position $(8, 13)$ is to the left of the main diagonal—by symmetry this will correspond to a lower P-position in column 13. The second bit in w is '0'. This means that the next upper P-position is in column 14. Then, by rules of game, it has to be at least in row 16, which indeed will be attained, so that the next P-position will be $(9, 16)$. In fact, by (26), the rows 14 and 15 cannot have P-positions to the left of the main diagonal, so that a prefix of the translate $D(w)$ is '1001'. Similarily, up to the last P-position of this word $(12, 18)$ we extend the prefix to '1001011' as is easely verified in the figure. The next upper P-position will be at least in row 22 since the least unused diagonal is $22 - 13 = 9$. With notation as above, here $m = 5$ and $l = 4$. (It will in fact be in row 23 since the next P-position is below the main diagonal.) After this a new perfect sector will start. This gives the last two '0's in the translate, '100101100', which may now be concatenated at the end of the first part of the bit-string, '00100', so that the new bit-string becomes '00100$\underline{1}$00101100'.

The next word, to be included to the dictionary, will be the underlined '1' (by symmetry a lower P-position) which translates to '0' since no upper P-position can belong to row 22 since column 13 is occupied by a lower P-position and again by (26). Then, the right picture in Figure 7 reveals how the next word to be included to the dictionary will be detected as '0010110' and translated to '10010011000'. Again, concatenating the new translates at the end of the existing strings give '0010010010110$\underline{0}$010010011000', where the underlined '0' is where the next word starts, and so on.

3.3 Maharaja Nim's dictionary

Maharaja Nim's dictionary is

$$1 \to 0 \tag{27}$$

$$01 \to 100 \tag{28}$$

$$00100 \to 100101100 \tag{29}$$

$$00110 \to 10010100 \tag{30}$$

$$000100 \to 10010110100 \tag{31}$$

$$001110 \to 100100100 \tag{32}$$

$$0010110 \to 10010011000 \tag{33}$$

$$00000100 \to 100101100111000 \tag{34}$$

$$000010010 \to 1001001111000100 \tag{35}$$

$$0000000 \to 10010110110100 \tag{36}$$

$$0010100 \to 100100110100 \tag{37}$$

$$0011110 \to 1001000100 \tag{38}$$

$$00000010 \to 100101101100100 \tag{39}$$

$$00001000 \to 100100111100100. \tag{40}$$

By computer simulations we have verified that each one of the words (27) to (35) does appear in Maharaja Nim's bit-string. We have included the code the Appendix, Section B. By our method of proof, we have found no way to exclude the latter five, but a guess is that they do not appear. At least they do not appear among the first 20000 bits of the bit-string. The following result gives the first part of the Main Theorem.

Lemma 3.2 (Completeness Lemma)**.** *When we read from Maharaja Nim's bit-string each prefix is contained in our extended dictionary of (left hand side) words of Maharaja Nim.*

Proof. Let us present a list in lexicographic order of all words in our extended dictionary together with the words we need to exclude:

0000000 → 10010110110100

00000010 → 100101101100100

00000011 'to exclude' (a)

00000100 → 100101100111000

00000101 'to exclude' (b)

0000011 'to exclude (c)'

00001000 → 100100111100100

000010010 → 1001001111000100

000010011 'to exclude' (d)

0000101 'to exclude' (e)

000011 'to exclude' (f)

000100 → 10010110100

000101 'to exclude' (g)

00011 'to exclude' (h)

00100 → 100101100

0010100 → 100100110100

0010101 'to exclude' (i)

0010110 → 10010011000

0010111 'to exclude' (j)

00110 → 10010100

001110 → 100100100

0011110 → 1001000100

0011111 'to exclude' (k)

01 → 100

1 → 0

This list is complete in the sense that any bit-string has precisely one of the words on the left hand side as a prefix. This motivates why it suffices to exclude the words 'to exclude'. For example (a) needs to be excluded since the only word in our list beginning with '0000001' continues with a '0'. Neither can we translate words beginning with '000001' continuing with '01' or '1'. This motivates why we need to exclude (b) and (c). All left hand side words in our dictionary beginning with 4 '0's continues with 100, which motivates that (e) and (f) need to be excluded, and so on. We move on to verify that the strings (a) to (k) are not contained in the bit-string.

No translate contains more than three consecutive '0's. To get a longer string one has to finish off one translate and start a new. The only translate starting with '0' is '0'. Thus, when a sequence of four or more '0's is interrupted it means that a new translate has begun. But all translates that begin with a '1' begins with '100'. Thus, a sequence of 4 or more '0's cannot be followed by '11' or '101'. This gives that the exclusion of the words (a),(b), (c), (e) and (f) is correct.

For the same reason, the string '100' in (d) has to be the prefix of some translate. Since the next two bits are '11', by the dictionary, this translate has to be '100'. But then the next translate has the prefix '11', which is impossible.

For the exclusion of (g) and (h), notice that any time three consecutive '0's appears, within a translate or between two translates, they are followed by the string '100'. Therefore, a string of three '0's cannot be followed by '11' or '101'.

For (i), notice that the sub-string '101010' is not contained in any translate. If it were, it needed to be either at the beginning of a translate, which is impossible (since all of them except '0' begin with '100') or be split between two. The latter is impossible since all translates except '0' ends with '00'. In analogy to this, also (j) must be excluded and similarly for (k) since no translate contains 5 consecutive '1's and all translates ends in a '0', but starts with either '0' or '10'.■

Proof of Theorem 1.1. By Lemma 3.2, our dictionary is correct. Since the left hand side words have at most 7 '0's we adjoin at most 6 P-positions in a sequence with $b_n - a_n$ distinct from n. Namely, by Lemma 3.1, when we start a new perfect sector we know that the next P-position will satisfy $b_n - a_n = n$. The number of bits in a translate is bounded (by 16) so that b_n can never deviate more than a bounded number of positions from $a_n + n$. Hence, by Proposition 2.1, the conditions of Lemma 2.3 are satisfied with the a-sequence as x, the b-sequence as y and $\delta = 1$, that is $b_n - a_n - n$ is $O(n)$ (as is also discussed in the paragraph before Lemma 2.3) which concludes the proof of Theorem 1.1. ■

By inspecting the dictionary one can see that, in fact, for all n, $-4 \le b_n - a_n - n \le 3$. Given this tight bound, the result in this section is quite satisfactory, but for the two gamesters trying to figure out how to quickly find safe positions, it does not quite suffice. The following question is left open.

Question 1. *Does Maharaja Nim's* decision problem*, to determine whether a given position* (x, y)*, with input length* $\log(xy)$*, is P, have polynomial time complexity in* $\log(xy)$*?*

We resolve this question for a similar game in [LW].

4 Dictionary processes and undecidability

Let us briefly discuss a problem related to the method used in this paper. Given a dictionary (of binary words and translations) and a starting string, will the translation process of the bit-string terminate?

More precisely, let us assume that we have a finite list of words $A = \{A_1, A_2, \ldots, A_m\}$ with translates $B_1, B_2, \ldots, B_m$ respectively, each word being a string of '0's and '1's, and where we, for simplicity, assume that none of the words in A is a prefix of another.

Take any finite bit-string S as a starting string (for example A_1 but it could be an arbitrary string, not necessarily in the list). A *read head* '_' starts to read S from left to the right and as soon as it finds a string A_i in A it stops, sends a signal to a printer at the other end which concatenates the translation B_i at the end of S. Then the read head continues to read from where it ended until it finds the next word in A, its translation being concatenated at the end, and so on.

If the read head gets to the end of the string without finding a word in the list A, the process stops with the current string as output. Otherwise, the process continues and gives as output an infinite string.

It follows from E. Post's *Tag Productions* [Mi1961, MC1964, Po1943] that it is algorithmically undecidable whether our dictionary processes stop or not. Let us include the very short proof. (Another proof is available in an extended version of this paper [LW].)

Theorem 4.1. *It is undecidable whether a given (prefix-free) dictionary process on a given initial string terminates.*

Proof. Let S be a finite string of letters from the alphabet $A = \{a_1, \dots a_n\}$ and let W be an associated list of words on this alphabet, say $w_1, \dots, w_n$. By [MC1964, page 2] it is undecidable whether the following Tag Production terminates. We read the first letter of S, say a_i, then erase the first two letters from S, and at last attach the word w_i at the end of S. Continue by performing the same operation on the resulting string. This production terminates if and only if, at some stage, the string consists of at most one letter.

Thus, to prove that it is undecidable whether our dictionary process terminates, it suffices to simulate this Tag system. By using binary words, each of length precisely $\lceil \log_2 n \rceil$ (fill out with zeros if needed), we can code each $a_i \in A$ as the binary representation of $i-1 \in \{0, \dots, n-1\}$. Our dictionary will consist of all left hand side words of the form $(i-1)x$, with $i-1, x \in \{0, \dots, n-1\}$ represented in binary, of length precisely $2\lceil \log_2 n \rceil$. (That is x is concatenated to the right of $(i-1)$.) The x will correspond to the letter in A that the tag production erases. In the dictionary process it will obviously not be erased. Rather, via our translation rules it will be ignored and the read head will be placed on the bit immediately to the right of its last digit. Each translation in the Dictionary will be chosen to interpret only to the first $\lceil \log_2 n \rceil$ letters. The translates for the dictionary process, to be concatenated to the right of the existing string, will be the corresponding binary interpretation of each word $w_i \in W$; say if $w_2 = a_1 a_5 a_8$ with $n = 8$ then, for any x, the dictionary's corresponding translate is '001x'$\to$ '000 100 111'. Therefore this dictionary process will terminate if and only if the tag production will. ■

There are infinitely many relatives to Maharaja Nim of the form: adjoin a (finite) list L of move options to Wythoff Nim, $(l_1, l_2) \in L$ if and only if $(x, y) \to (x - l_1, y - l_2)$ is a legal move, for all positions (x, y) such that $x - l_1 \geq 0$

and $x - l_2 \geq 0$. It is easy to see that Proposition 2.1 and (2) hold also for these extensions of Wythoff nim (provided that L is finite). For any given such generalization, is it possible to determine the greatest departure from n for $b_n - a_n$? Even simpler, is it decidable, whether there is a P-position above some straight line?

Question 2. *Given the moves of Wythoff Nim together with some finite list L of moves, that is ordered pairs of integers (in Maharaja Nim the list is $\{(1,2),(2,1)\}$), and a linear inequality in two variables x and y, is it decidable whether there is a P-position in the game which satisfies the inequality?*

Of course, it is not even clear whether a given finite generalization of Maharaja Nim (with $(l_1, l_2) \in L$ if and only if $(l_2, l_1) \in L$) will produce a finite dictionary in the sense of Section 2; see also [LW], where we study a similar but *non-prefix free dictionary* for the game where $L = \{(2, 3, (3, 2)\}$.

Appendix

A The Maple code corresponding to Figure 2

The following code includes the P-positions of both Wythoff Nim and Maharaja Nim in one and the the same diagram.

```
restart: with(plots): with(plottools):

N:=50;

theLine1:=CURVES([[0.0,0.0], [evalf(N), evalf(N*(1+sqrt(5))/2)]]):
theLine2:=CURVES([[0.0,0.0], [evalf(N*(1+sqrt(5))/2), evalf(N)]]):

#Compute the P-positions of Wythoff Nim and store as a list of squares.
#0=Not yet computed, 1=P, 2=N.
for i from 0 to N do for j from 0 to N do A[i,j]:=0: od: od:
for i from 0 to N do for j from 0 to N do if A[i,j]=0 then A[i,j]:=1:
for k to N do A[i+k, j]:=2: A[i+k,j+k]:=2: A[i,j+k]:=2: od: fi: od: od:
rectListW:=[]: for i from 0 to N do for j from 0 to N do if A[i,j]=1
then rectListW:=[op(rectListW), [[i,j],[i,j+1],[i+1,j+1],[i+1,j]]]: fi:
```

```
od: od:

#Draw the P-positions and the two lines with slopes the golden ratio:
display(polygonplot(rectListW, color=red), theLine1, theLine2, axes=none,
scaling=constrained, view=[0..N, 0..N]);

#Compute the P-positions of Maharaja Nim:
for i from 0 to N do for j from 0 to N do A[i,j]:=0: od: od:
for i from 0 to N do for j from 0 to N do if A[i,j]=0 then A[i,j]:=1:
A[i+1,j+2]:=2:
A[i+2,j+1]:=2:
for k to N do A[i+k, j]:=2: A[i+k,j+k]:=2: A[i,j+k]:=2: od: fi: od: od:
rectListM:=[]: for i from 0 to N do
for j from 0 to N do if A[i,j]=1 then rectListM:=[op(rectListM),
[[i+0.2,j+0.2],[i+0.2,j+0.8],[i+0.8,j+.8],[i+0.8,j+0.2]]]: fi: od: od:

display(polygonplot(rectListM, color=blue), axes=none, scaling=constrained);
display(polygonplot(rectListM, color=blue),
polygonplot(rectListW, color=red), theLine1, theLine2, axes=none,
scaling=constrained, view=[0..N, 0..N]);
```

B The Maple code for Maharaja Nim's dictionary.

This code explores whether the first 9 words in Maharaja Nim's dictionary suffice.

```
dictionary:={[1], [0,1], [0,0,1,0,0], [0,0,1,0,1,1,0], [0,0,1,1,0],
[0,0,0,1,0,0],[0,0,0,0,1,0,0,1,0], [0,0,0,0,0,1,0,0], [0,0,1,1,1,0]};

translation:=table([[1]=[0], [0,1]=[1,0,0], [0,0,1,0,0]=[1,0,0,1,0,1,1,0,0],
[0,0,1,0,1,1,0]=[1,0,0,1,0,0,1,1,0,0,0],[0,0,1,1,0]=[1,0,0,1,0,1,0,0],
[0,0,0,1,0,0]=[1,0,0,1,0,1,1,0,1,0,0],
[0,0,0,0,1,0,0,1,0]=[1,0,0,1,0,0,1,1,1,1,0,0,0,1,0,0],
[0,0,0,0,0,1,0,0]=[1,0,0,1,0,1,1,0,0,1,1,1,0,0,0],
[0,0,1,1,1,0]=[1,0,0,1,0,0,1,0,0]]);
```

```
theString:=[0,0,1,0,0]: reader:=0:
for times to 12000 do foundWord:=false:
for i to 9 do if not foundWord then theWord:=theString[reader+1..reader+i]:
if member(theWord, dictionary) then foundWord:=true:
theString:=[op(theString), op(translation[theWord])]:
reader:=reader+i: fi: fi: od: if not foundWord
then print(reader, theString[reader+1..reader+20]): fi:
if times mod 100 = 0 then print(times, nops(theString)): fi: od:
```

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