Abstract

Interest in 2-player Impartial games often concerns the famous theory of Sprague-Grundy. In this thesis we study other aspects, bridging some gaps between Combinatorial Number Theory, Computer Science and Combinatorial Games. The family of heap games is rewarding from the point of view of Combinatorial Number Theory, partly because both the positions and the moves are represented simply by finite vectors of nonnegative integers. For example the famous game of Wythoff Nim on two heaps of tokens has a solution originating in Beatty sequences with modulus the Golden Ratio. Sometimes generalizations of this game have similar properties, but mostly they are much harder to grasp fully. We study a spectrum of such variations, and our understanding of them ranges from being complete in the case of easier problems, to being very basic in the case of the harder ones. One of the most far reaching results concerns the convergence properties of a certain $\star\star$-operator for invariant subtraction games, introduced here to resolve an open problem in the area. The convergence holds for any game in any finite dimension. We also have a complete understanding of the reflexive properties of such games. Furthermore, interesting problems regarding computability can be formulated in this setting. In fact, we present two Turing complete families of impartial (heap) games. This implies that certain questions regarding their behavior are algorithmically undecidable, such as: Does a given finite sequence of move options alternate between N- and P-positions? Do two games have the same sets of P-positions? The notion of N- and P-positions is very central to the class of normal play Impartial games. A position is in P if and only if it is safe to move there. This is virtually the only theory that we need. Therefore we hope that our material will inspire even advanced undergraduate students in future research projects. However we would not consider it impossible that the universality of our games will bridge even more gaps to other territories of mathematics and perhaps other sciences as well. Classical game theory has lately become very popular, due to its impact on for example, psychology, economics and biology. Combinatorial games have so far shown a more theoretical beauty, apart from the fact that the field is also concerned with solutions of popular recreational games, games that people love to play.

Impartial Games and Recursive Functions

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1 Papers

The thesis consists of 8 papers.

Paper 1: 2-pile Nim with a Restricted Number of Move-size Imitations (with an appendix by Peter Hegarty); also Section 3.1.

Paper 2: Blocking Wythoff Nim; also Section 3.2.

Paper 3: A Generalized Diagonal Wythoff Nim; also Section 3.3.

Paper 4: Maharaja Nim, Wythoff's Queen meets the Knight (with Johan Wästlund); also Section 3.4.

Paper 5: Invariant and dual subtraction games resolving the Duchêne-Rigo conjecture (with Peter Hegarty and Aviezri S. Fraenkel); also Section 3.5.

Paper 6: The $\star$-operator and invariant subtraction games; also Section 3.6.

Paper 7: From heaps of matches to the limits of computability (with Johan Wästlund); also Section 3.7.

Paper 8: Impartial games emulating one-dimensional cellular automata and undecidability; also Section 3.8.

Papers 1,2,3,5 and 6 are published as described in the bibliography. The other three have been submitted to journals for peer reviewing (Paper 8 has recently been accepted). My coauthors are Johan Wästlund (Papers 4 and 7) and Peter Hegarty, Aviezri S. Fraenkel (Paper 5). P. Hegarty has also contributed an Appendix for Paper 1.

A fairly straightforward, but rough, way to categorize the papers are as follows: Papers 1 and 2 concern blocking maneuvers, including a move-size dynamic variation, of Wythoff Nim. Papers 3 and 4 concern natural extensions of Wythoff Nim that 'adjoin moves' to those of Wythoff Nim. Papers 5 and 6 concern a certain $\star$-operator of (vast generalizations of classical) invariant subtraction games. Papers 7 and 8 concern Turing completeness of two families of impartial (heap) games.

"A problem worthy of attack, proves its worth by fighting back!"

I got this proverb from Aviezri Fraenkel some years ago. It kept me busy a couple of years.

2 A brief introduction

We study interconnections between *combinatorial games* [C1976, BCG1982], *number theory* and *computer science*, with an emphasis on the former. A game consists of a finite set of *positions*, a *ruleset*, two *players* alternating moves and a declaration of who begins. Thus, given any position, the move options are also known; if no move is available, the game is *terminal* and the player who is *not* in turn to move is declared the *winner*.

Example 1. *The Game of "21" is a popular children's game. Two players alternate in subtracting one of the numbers 1 or 2 from a given non-negative integer, starting with* 21 *until the position is* 0. *If both play optimally, will the First or the Second player win?*

We will return to this example in a moment. Sometimes very simple rules can make a game nearly trivial, other times interesting patterns emerge; yet other games can exhibit never ending complexities. Of course we all know that the game of Chess has a simple ruleset but is very hard to master. The famous game of Go is another such example of even greater complexity. But we will later show that even by restricting us to so-called *impartial heap games*, many interesting problems will withstand arbitrarily sophisticated computations.

In our setting, the ruleset is the same for both players. That is, given any position, the set of options is the same no matter whose turn it is to move. Thus we study the family of so-called *impartial games*. For example the game of Chess is not impartial; in general terms, the rules are similar for both players, but if we look at a given position, the move options are usually very different depending on whether Black or White is the current player.

By our axioms, each game position has finitely many options (as in Chess) and terminates within a finite number of moves (Chess might not).

There is no hidden information such as in Whist or Poker; neither is there any "random element" such as a dice (nor is there any influence of psychology or such unpredictable conditions). There is an important implication of this: the *outcome* of a game can always be computed in finite time.

There is some standard terminology for the family of impartial games, but not much is needed. For the *outcome* of a game, we denote by N any position from which the *next player* (the player in turn to move) wins. Any other position is denoted by P, the *previous player* wins. Since a game is finite, there will be no infinite loops or ties, hence the sets of N- and P-positions partition the set of positions of a given game. Clearly, any terminal position is in P. Then all positions that have moves to terminal positions can be listed. They will all be labeled N. Any position not yet listed, that has only listed N-positions as options, must belong to the set of P-positions, and so on. In general, any position that has a move to some P-position is in N, and otherwise, if no P-position is available, it is in P.

By this simple deterministic algorithm we see that our games belong to the class of so-called *perfect information* games: given any position and sufficient computational power it is always possible to compute a winning move, if there is one. In fact, all combinatorial games belong to this family where absolute knowledge is possible.

On the other hand, in a game of Yatzy, for example, the meaning of "optimal play" is completely different, since absolute knowledge is not possible. A reasonable guide would be to compute the expected score of all options and follow the maximal expectation. Even if you play at best from a favorable position in Yatzy there is in general a positive probability that you will lose. If you start with 2,2,3,6,6 and three- and four of a kind were the only remaining slots, then everybody would agree that we save 6,6 and throw three dices. But what if the two remaining throws both produce 2, 2, 1? You play optimally, but lose 8 points (if keeping the 2s would have resulted in the same throws). Games of perfect information don't exhibit such behavior.

We will often restrict our attention to a subfamily of all impartial games, the so-called *heap games*. A position consists of a finite number of *heaps* (or *piles*) each with a finite number of *tokens* (or *matches* etc). Various subclasses of such games are often called "take-away" games [G1966, S1970, Z1996], whereas games such as "21" are often called *subtraction games*.

Example 2. *Let us examine the winning strategy for the game of "21" in Example 1, interpreted as a heap game. Since the unique terminal position is "0", a player who can move to a pile with three tokens wins. This follows since* $3 - 1 = 2$ *and* $3 - 2 = 1$*. Now, by the same argument a pile with 6 tokens is losing and in general all heap sizes divisible by 3. Hence the game of "21" is losing for the First player, it is a P-position.*

For heap games, we often consider a game's enumerably infinite set of all possible starting positions. In particular it will allow us to explore the famous territory of the Church-Turing Thesis concerning recursive or computable functions.

In the 17-18th century W. G. Leibniz asked whether it is possible to build a mechanical device that can test any mathematical proposition's accuracy. This later became known as Hilbert's Entscheidungsproblem, which asks for an algorithm to decide whether a given statement is provable from the axioms of first order logic. A similar problem is to try to invent an algorithm that evaluates whether two given propositions are equivalent. In the 1930s Turing and Church independently proved that such algorithms do not exist. Turing's approach was to reduce the *halting problem* of his universal "machine" to the Entscheidungsproblem. He had already established that the halting problem is algorithmically undecidable: there is no Turing machine that can take as input the code of another Turing machine and decide whether it will halt or not for a given input. Both the code that describes the machine and its input are finite. Thus, if we want to reduce some problem in our setting of impartial games to the halting problem we must ensure that the "code", whatever we use, is finitely described, and also by the Church-Turing thesis, if it produces an "output" in finite time, it must be correct. If it does not "halt" (which will have different meanings in our setting), it must not produce an incorrect result. Today there is a spectrum of abstract or even real machines capable of universal computation, such as all everyday computers, universal Turing machines [T1936], Posts Tag productions [M1961, P1943] and some families of Cellular Automata [N1963, HU1979, W1984a, W1984b, W1984c, C2004] and many more. (In either case there is no limit in time or memory used.) We will return to these type of problems in Papers 7 and 8.

In the previous discussions many rulesets were 'finite'. Before we enter the next section with the summary of our papers, let us mention two more examples of classical impartial games with particularly simple, yet infinite, rulesets.

Example 3. *[B1902] k-pile Nim, for k a positive integer. Positions: k heaps of a finite number of tokens. Moves: remove any positive number of tokens from precisely one of the piles, at most a whole pile. P-positions: The nim-sum of the heap sizes (addition modulo 2 of the binary expansions of the heap sizes) equals zero; in case* $k = 2$ *this means that the 2 piles have the same number of tokens.*

Example 4. *[W1907] Wythoff Nim. Positions: 2 heaps of a finite number of tokens. Moves: As 2-pile Nim but also a "diagonal" type of move, remove the same number of tokens from both piles as long as each remaining pile contains a non-negative number of tokens. Thus, the game can equivalently be played on a large Chess board with one single piece which moves as the Queen, but with the restriction that, by moving, the "Manhattan distance" to position* $(0, 0)$ *must decrease. P-positions: Let* $\phi = \frac{1+\sqrt{5}}{2}$ *denote the golden ratio. The P-positions are of the forms* $(\lfloor \phi n \rfloor, \lfloor n\phi^2 \rfloor)$ *and* $(\lfloor \phi^2 n \rfloor, \lfloor n\phi \rfloor)$ *for all non-negative integers* n*. They are displayed in Figure 1 to the right.*

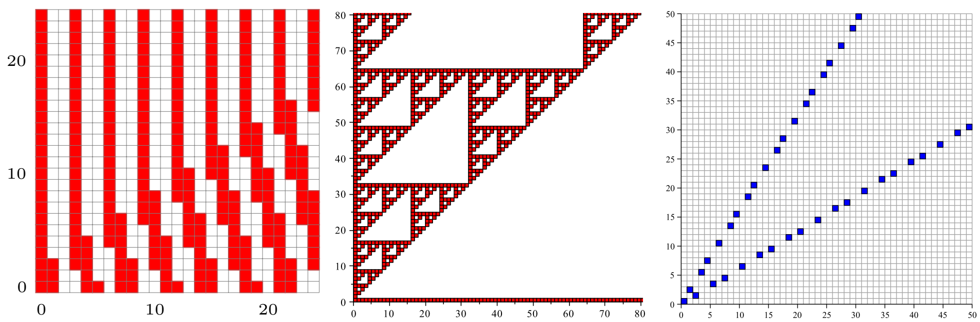

Figure 1: What do we see in a pattern? Can we imagine an infinite continuation? How would you generate these three 2-dimensional patterns? The leftmost contains two periodic sectors, the other two exhibit well known fractal or self-similar behavior. They can be generated in a few different ways. In this thesis we show that such patterns sometimes can be interpreted as solutions of Combinatorial Games on heaps of tokens and/or matches. There are two overarching research questions in this thesis. (1) What patterns do combinatorial games with given rules produce? (2) Given a pattern, can we find a combinatorial game that produces it? The leftmost pattern is described in Paper 7, the middle one is Pascal's triangle modulo 2 concerning Papers 7 and 8, whereas the rightmost pattern is explained in Example 4.

3 Overview of papers

Much inspiration for this thesis came from our three initial examples of impartial take-away games, with a certain emphasis on variations of Wythoff Nim. A major theme is to construct new games out of known ones: Papers 1, 2, 3, 4, 5 and 6.

Another theme is to emulate or mimic other interesting mathematical structures and sequences by the P-positions of a heap game: Papers 4, 5 and 8. In this context we also investigate whether apparently different games can have the same sets of P-positions: Papers 1 and 7. In fact, such questions are demonstrated algorithmically undecidable in Paper 7.

Exploiting the similarities of the descriptions of positions and move options of our heap games, surprising properties arise. In this context, the $\star$-operator is introduced in Papers 5 to resolve a recent conjecture of Duchêne and Rigo [DR2010]. In Paper 6 we investigate further properties of this operator.

We demonstrate that machines capable of universal computation, can be emulated, via sequences of P-positions, by certain impartial heap games; Papers 7 and 8. For the latter case a so-called *move-size dynamic* rule is used, where the options for the next move depend on the previous move. We introduce a variation of this already in our first paper.

3.1 Imitation Nim and Wythoff's sequences, Paper 1

This paper concerns a variation of the classical game of Nim on two piles as in Example 3. Suppose that the previous player removed x tokens from the smaller heap (any heap if they have equal size). Then the next player may not remove x tokens from the larger heap. We call this game *Imitation Nim* (although, as remarked by Aviezri Fraenkel the first time I met him, *L*imitation Nim would also have been an appropriate name). Notice that by this maneuver, the winning strategy of 2-pile Nim is altered. For example, the player who moves from the position $(1, 1)$ will lose in Nim, but win in Imitation Nim. It turns out that the P-positions correspond to those of Wythoff Nim, Example 4. The game generalizes nicely. Suppose that $m - 1$ consecutive imitations from one and the same player are allowed, but not the mth one. For example, with $m = 2$ and $0 < x \leq y$, suppose that the three most recent moves were $(x, y) \to (x - z, y) \to (x - z, y - z) \to (x - z - w, y - z)$, alternating between the two players. Then precisely the move to $(x - z - w, y - z - w)$ is prohibited. The P-positions of this generalization of Imitation Nim correspond to those of a variation of Wythoff Nim with a so-called *blocking maneuver* on the diagonal options, studied first in [HL2006].

3.2 Blocking Wythoff Nim; searching for possible Beatty sequences, Paper 2

Let us begin by giving some background to this paper. Let m be a positive integer. In the variation of Wythoff Nim with a blocking maneuver proposed in [HL2006], the previous player may, before the next player moves, block off $m - 1$ of the diagonal type options and declare them forbidden. After the next player has moved, any blocking maneuver is forgotten. It turns out that the P-positions of these generalizations have similar structures as those of Wythoff Nim (Example 4). The sequences of their coordinates approximate very closely straight lines with irrational slopes on the 2-dimensional integer lattice. Namely the sequences of the coordinates can be approximated by so-called homogeneous Beatty sequences $(\lfloor n\alpha \rfloor)$, where α is a positive irrational and n ranges over the positive integers. (This was proved independently by Hegarty in [L2009, Appendix] and Fraenkel Peled in [FP].)

Two sets of positive integers are called *complementary* if each positive integer occurs in precisely one of them. It is a well known result [B1926] that the sets $\{\lfloor n\alpha \rfloor\}$ and $\{\lfloor n\beta \rfloor\}$, where n ranges over the positive integers, are complementary if and only if α, β are positive irrationals satisfying $\alpha^{-1} + \beta^{-1} = 1$. A special case of this is given in Example 4. Combinatorial games with a blocking maneuver, or so-called Muller Twist, were proposed via the game Quarto in "Mensa Best Mind Games Award" in 1993. Later the idea appeared in the literature [HR2001, SS2002].

Having observed, in [HL2006], that a blocking maneuver on the diagonal type moves gives rise to interesting sequences of integers, I set out to study two other natural variations of Wythoff Nim with a blocking maneuver [L] and Paper 2, where at most a given finite number of options can be blocked at each stage of the game; in [L] blocking is allowed exclusively on the Nim-type options, whereas in the included paper blocking is allowed on all options of Wythoff Nim.

An exact formula for the P-positions is given for the blocking variation of Wythoff Nim, where at most one (non-restricted) option may be blocked. For this game, the *upper* P-positions have *split* into two sequences of P-positions, one with slope ϕ (similar to the formula for Wythoff Nim) and the other with slope 2. A position (x, y) is *upper* if $y \geq x$. We understand the P-positions for two or fewer blocked off options and conjecture results for some greater blocking parameters. The general problem seems very hard. (In contrast, the P-positions of the games in [L] can be described via Beatty sequences for all blocking parameters, generalizing A. S. Fraenkel's classical p-Wythoff Nim [F1982].)

A last remark regarding this paper is that the family of comply games we discuss are not of the form considered in [S1981], where it is proved that for all impartial games in consideration, almost all positions are next player winning. In the same sense, our comply maneuver can be applied to any impartial game, labeling almost all positions as previous player win. The reason for this is that, at each stage of game, the previous player has to present a non-empty set for the next player's consideration. In a sense, this turns the usual rules inside-out.

3.3 A Generalized Diagonal Wythoff Nim and splitting beams of P-positions, Paper 3

The behavior of a possible *splitting* of sequences of P-positions into two sequences of distinct slopes is discussed also in this paper. The P-positions of Nim lie on the single beam of slope 1, whereas those of Wythoff Nim lie on the beams of slopes ϕ and ϕ^{-1}. Therefore, going from Nim to Wythoff Nim has split the single *beam* of P-positions in Nim into two new P-beams for Wythoff Nim of distinct slopes. Let p, q be positive integers. If we adjoin, to the game of Wythoff Nim, new moves of the form (pt, qt) and (qt, pt), for all positive integers t, will the upper P-positions of the new game, denoted (p, q)-GDWN, split once again into two new distinct slopes? Here we prove that the ratio of the coordinates of the upper P-positions of this game do not have a unique accumulation point if $p = 1$ and $q = 2$. Via experimental results we conjecture that the upper P-positions of a (p, q)-GDWN game split if and only if (p, q) is either a Wythoff pair or a dual Wythoff pair, that is of the form $(p, q) = (\lfloor \phi n \rfloor, \lfloor n\phi^2 \rfloor)$ or $(\lceil \phi n \rceil, \lceil n\phi^2 \rceil)$, for n a positive integer. In a recent preprint [L1], which is not included in this thesis, I prove that $(1, 2)$-GDWN splits. Two new discoveries made this possible.

Lemma 5. *Let $\{(x_i, y_i)\}$ define the set of upper P-positions of some* extension *of Wythoff Nim (for all* i, $x_i \le y_i$ *and* $x_i < x_{i+1}$*). Then, for all positive integers* n,

$$\liminf_{n\to\infty} \frac{\#\{i > 0 \mid x_i < n\}}{n} \ge \phi^{-1}.$$

Lemma 6. *If there is a positive lower asymptotic density of x-coordinates of P-positions above the line* $y = 2x$*, then the upper P-positions* $\{(a_n, b_n)\}$ *of* $(1,2)$*-GDWN split.*

In [L1], we show that $(1,2)$-GDWN satisfies the hypothesis of Lemma 6 and that the first result implies the second.

The conjecture is that there are precisely two accumulation points for the upper P-beams, namely to the ratio of coordinates $1.477\ldots$ and $2.247\ldots$ respectively. Another related research project is [L3].

3.4 Maharaja Nim and a dictionary process, Paper 4

In this paper, coauthored with J. Wästlund, we study an *extension* of Wythoff Nim, where the Queen and Knight of Chess are combined in one and the same piece, the *Maharaja* (still no coordinate increases by moving). The game is called *Maharaja Nim*. One can also view this game as a *restriction* of $(1,2)$-GDWN. It is clear that the P-positions of Wythoff Nim will be altered for this game, namely the "smallest" P-position of Wythoff Nim are $(1,2)$ and $(2,1)$, constituting precisely the new move options introduced for Maharaja Nim. However we have managed to prove that the P-positions remain within a bounded distance of the lines ϕ and ϕ^{-1}. To obtain such a result we have used an unconventional method in this field, namely a certain dictionary process on binary words and translations, a process that we also prove is in general undecidable in the appendix. We also generalize an already very nice result from [FP] to a "Central Lemma" in our paper.

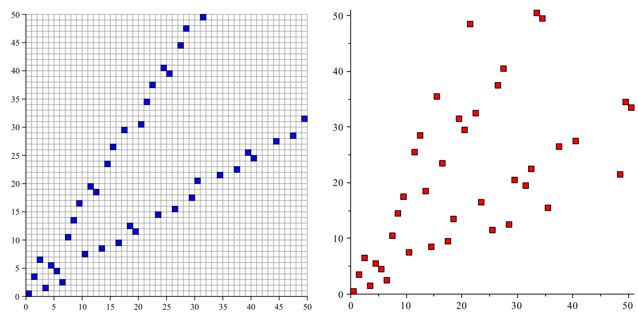

Figure 2: The first few P-positions of Maharaja Nim and $(1, 2)$-GDWN respectively. How regular are these games?

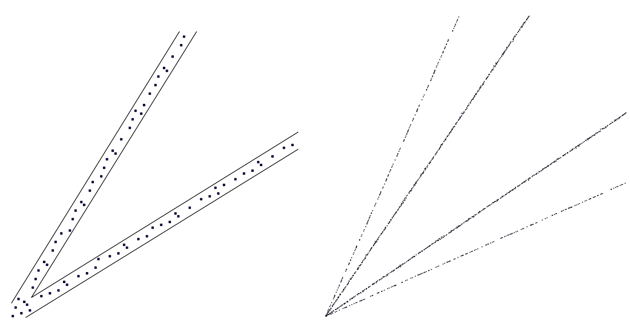

Figure 3: The leftmost figure indicates that we are able to capture the behavior of Maharaja Nim's upper P-positions within a narrow stripe of slope ϕ. On the other hand, we have proved that the P-positions of GDWN to the right will eventually depart from any such stripe, however wide we make it. Extensive computations make us believe that perhaps the upper pair of P-beams' slopes will converge to the accumulation points $1.478\ldots$ and $2.248\ldots$ respectively. See also [L1] for a proof of an actual split of the upper P-beams, a result that we have chosen not to include in this thesis because it is in the process of being peer reviewed.

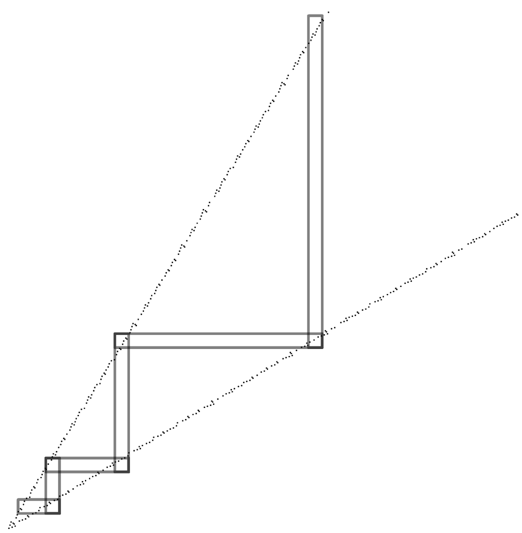

Figure 4: Is it possible to decide in polynomial time, whether a given position is in P for Maharaja Nim? A "telescope" with focus $O(1)$ and reflectors along the lines ϕn and n/ϕ attempts to determine the outcome (P or N) of some position, (x, y) at the top of the picture. The method is successful for a similar game called $(2, 3)$-Maharaja Nim [L4]. (It gives the correct value for all extensions of Wythoff Nim with a finite non-terminating converging dictionary). The focus is kept sufficiently wide (a constant) to provide correct translations in each step. The number of steps is linear in $\log(xy)$.

3.5 Invariant games, the $\star$-operator and complementary Beatty sequences, Paper 5

This paper is joint with P. Hegarty and A.S. Fraenkel. An *invariant subtraction game* can be identified with a *move set* $\mathcal{M}$ of k-tuples of non-negative integers (not all zero). Given a position, that is another k-tuple of non-negative integers $\boldsymbol{x} = (x_1, \ldots, x_k)$ (possibly all zero), a player may use any vector $\boldsymbol{m} = (m_1, \ldots, m_k)$ from the move set and subtract it from $\boldsymbol{x}$ to obtain the next position $\boldsymbol{x} \ominus \boldsymbol{m} = (x_1 - m_1, \ldots, x_k - m_k)$, provided $\boldsymbol{x} \succeq \boldsymbol{m}$, that is $x_i \geq m_i$ for each i. As usual a player unable to move loses. Examples 1, 3 and 4 belong to this class of games.

In this paper we resolve a conjecture from [DR2010]. They conjectured that, given a pair of complementary Beatty sequences (a_i) and (b_i) (as described in the second paragraph in Section 3.2), there is an invariant subtraction game for which the P-positions constitute precisely all the pairs (a_i, b_i) and (b_i, a_i), together with the terminal position $(0, 0)$.

We give a surprisingly simple solution to this problem. Namely take the description of the candidate P-positions as *moves* in another invariant subtraction game (without $(0, 0)$). Then the P-positions of the new game (without $(0, 0)$) correspond precisely to the moves of another invariant subtraction game, which has the original candidate set of P-positions as its set of P-positions. We denote the invariant subtraction game with moves corresponding to the P-positions by $\mathcal{M}^\star$, where $\mathcal{M}$ is the original game, and thus show that the relation $\mathcal{M} = (\mathcal{M}^\star)^\star$ holds. We extend the result to a somewhat larger class of 'super-additive' sequences. See Figure 5 for an example. The move sets in Papers 5 and 6 will have a different interpretation than those in Paper 7 (although the notation is the same). The reason is that it will be more natural to add integer vectors in the latter paper.

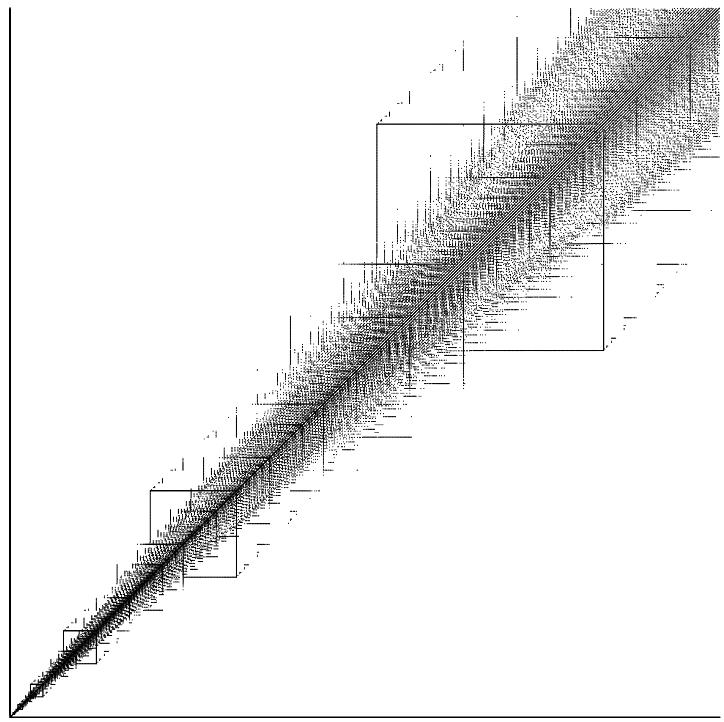

Figure 5: The picture illustrates the initial P-positions of the game (Wythoff Nim)*, or equivalently $(0,0)$, in the lower left corner, together with the moves of the game (Wythoff Nim)**. We have made computations to 12000 obtaining a similar behavior and proved some of it, but the overall pattern remains a mystery, although it is contained between half lines from the origin of slopes ϕ^{-1} and ϕ. In addition, a characterization of infinitely many log-periodic positions has been obtained in [L5]. This result is not included in this work. (Wythoff Nim)** appears to be a very complicated game to play, although it has precisely the same set of P-positions as Wythoff Nim. What is more, the former game has a very nice property, which is absent in Wythoff Nim, namely it is reflexive, that is (Wythoff Nim)**=(Wythoff Nim)^{2k*} for all $k \geq 1$.

3.6 Convergence of the $\star\star$-operator, Paper 6

Here we explain some basic properties of the $\star$-operator, from Paper 5, and call $\mathcal{M}^\star$ the *dual* of $\mathcal{M}$. Whenever $\mathcal{M} = \mathcal{M}^{\star\star} = (\mathcal{M}^\star)^\star$ holds we say that $\mathcal{M}$ is *reflexive*. We prove that $\mathcal{M}$ is reflexive if and only if the set

$$\{\boldsymbol{m}_1 \ominus \boldsymbol{m}_2 \succ \boldsymbol{0} \mid \boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{M}\}$$

is a subset of the set of N-positions, $\mathcal{N}(\mathcal{M})$, see Figure 6 for an example. We define the notion of *convergence* of a sequence of invariant subtraction games and prove that the *limit game* resulting from an infinite recursive application of the $\star\star$-operator on an invariant subtraction game exists. Many problems remain to be resolved, such as: find an explicit formulation of some limit game, without using the notion of a sequence of invariant subtraction games.

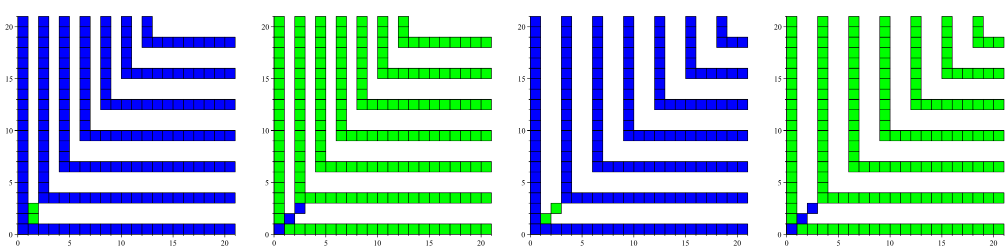

Figure 6: The figures illustrate three recursive applications of the $\star$-operator on $\mathcal{M} = \{(1,1),(1,2)\}$ for small positions. In the first figure the green squares represent the two moves in $\mathcal{M}$ and the repetitive blue pattern its (initial) set of P-positions, $\mathcal{P}(\mathcal{M})$; the next one illustrates the repetitive patterns in $\mathcal{M}^\star$ with its (finite) set of P-positions, $\mathcal{P}(\mathcal{M}^\star)$, and so on. The game $\mathcal{M}$ is non-reflexive since $(1,2) \ominus (1,1) = (0,1) \in \mathcal{P}(\mathcal{M})$. Neither is the dual, $\mathcal{M}^\star$, since $(1,0)$ and $(3,2)$ are moves, but $(3,2) \ominus (1,0) = (2,2) \in \mathcal{P}(\mathcal{M}^\star)$. On the other hand $\mathcal{M}^{\star\star} = \{(1,1)(2,2)\}$ is reflexive, since $(2,2) \ominus (1,1) = (1,1) \in \mathcal{M}^{\star\star} \subset \mathcal{N}(\mathcal{M}^{\star\star})$. Hence $\mathcal{M}^n$ is reflexive for all $n \geq 2$. We conjecture that this holds for any $\mathcal{M}$, with two or less moves, in any dimension.

3.7 Invariant heap games, cellular automata and undecidability, Paper 7

In this paper we discuss a family of heap games, played on k heaps of matches, with a finite number of invariant move options (generalizing Examples 1 and 2). Here the rules are relaxed so that, by moving, the *total* number of matches in all heaps must decrease, but the number may increase in individual heaps. We prove, by relating the P-positions of a game to the updates of one-dimensional Cellular Automata (CA), that it is algorithmically undecidable whether two games have identical sets of P-positions. In fact, we reduce this problem to that of determining whether a finite binary string "101" occurs in the update of the CA, a problem which is known to be equivalent to the halting problem of a universal Turing machine. The construction uses an injective map from one-dimensional cellular automata to a class of (non-invariant) so-called *modular games* on two heaps of matches, a "tape heap" and a "time heap". A given n-ary update function for the CA is simulated via specific move options that are legal from distinct congruence classes modulo n prescribed by the size of the time heap. The size of the tape heap simulates the position of the CA's tape whereas the size of the time heap, kn, simulates the kth update of the CA. The computation is carried out via the modular game's (binary) outcome function. Then, by introducing k more heaps, called the *gadget*, we emulate the modular games via a subset of invariant games on $k + 2$ heaps.

3.8 An impartial game on two heaps emulating the rule 110 CA, Paper 8

Inspired by the discoveries in Paper 7, we ask the following question: is it possible for an impartial heap game to encompass universal computation using only two heaps? In particular can one emulate directly some one-dimensional cellular automata for which many questions are known to be algorithmically undecidable? In this paper we discuss two constructions which emulate the *rule 110* CA, Figure 7, which was proved undecidable by Mathew Cook [C2004] (resolving a conjecture by Steven Wolfram). Our heap game variant in Figure 8 uses only two heaps and is similar to the game of Imitation Nim in that it takes advantage of a certain kind of move-size dynamics, which gives the heaps different meanings, simulating "time" and "space" respectively. We prove that the patterns of P-positions of our game are equivalent to the patterns in the update of the rule 110 CA and thereby many questions regarding our heap game are undecidable.

In fact, in this paper we define an infinite family of move-size dynamic take-away games on two heaps, including also the well-known patterns of Pascal's triangle modulo 2, corresponding to the cellular automaton with update function the "Xor gate" (rule 60 in Wolfram's notation). We also show how these games have nice interpretations as board games, see Figure 3.8.

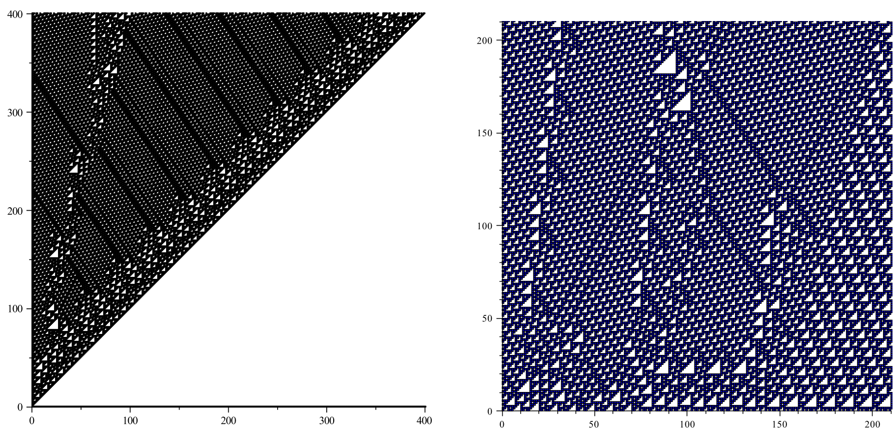

Figure 7: The leftmost picture is produced by the updates of the one dimensional Cellular Automaton rule 110, the central cell: $000 \to 0, 001 \to 0, 010 \to 1, 011 \to 1, 100 \to 1, 101 \to 1, 110 \to 1, 111 \to 0$. The initial one dimensional pattern is given by the doubly infinite initial string $\ldots 0011 \ldots$ (the black cells at the bottom are the 1s and time flows upwards). The origin of the name comes from the binary number 1101110 obtained by letting the time run downwards; the rule 124 CA is isomorphic to rule 110. The y-axis corresponds to "time" and the x-axis to "space". We have omitted the negative part of the string since this area will be covered by 0s. (Compare with the patterns of the much simpler rule 60 CA, leftmost in Figure 1.) The rightmost figure is produced by the same automaton with a somewhat more complicated initial string. Periodic "gliders" appear in a periodic background ether, interacting in unpredictable ways.

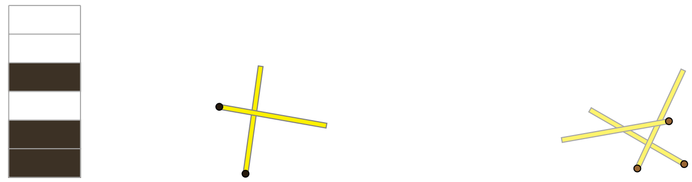

Figure 8: A rule 110 game variation on two heaps. The rightmost heap represents the previous player's removal of matches. The coloring of the tokens is essential; the last y matches can only be removed if the top y tokens are non-black. (Thus, if there are no tokens left, then the last match can always be removed.) The number of tokens a player can remove depends both on the current and the previous player's removal of matches. If the previous player removed m_p matches then $m-1 \leq t \leq m_p+m$ tokens can be removed together with $1 \leq m$ matches. This is provided that the terminal condition allows. Who wins the current game?

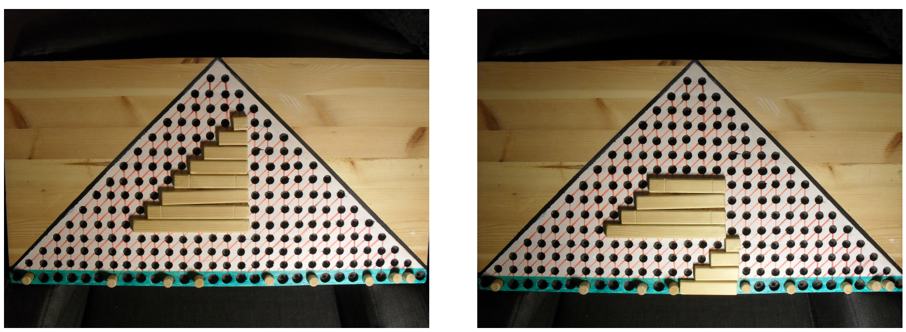

Figure 9: A board game variation of the rule 110 game, called the Triangle placing game, illustrating how the next player uses the only terminal move option from the given position. The top of the next right triangle must touch the base of the previous right triangle (strictly below). The pegs at the bottom prevents certain moves.

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