Imitation Nim

Urban Larsson, Göteborg, Sweden

The Stony Brook (satellite) workshop

July 18, 2013

Pedigree

We study impartial games played on heaps of tokens

Pedigree

We study impartial games played on heaps of tokens

- 2-pile Nim (Bouton 1902)

Pedigree

We study impartial games played on heaps of tokens

- 2-pile Nim (Bouton 1902)
- Wythoff Nim (Wythoff 1907, Fraenkel 1982)

applying standard variations of such games

Pedigree

We study impartial games played on heaps of tokens

- 2-pile Nim (Bouton 1902)
- Wythoff Nim (Wythoff 1907, Fraenkel 1982)

applying standard variations of such games

- move-size dynamics (Whinihan 1963, Schwenk 1970, Epp-Ferguson 1980, Zieve 1996, Holshouser-Reiter 2003, Gurvich 2012)

Pedigree

We study impartial games played on heaps of tokens

- 2-pile Nim (Bouton 1902)
- Wythoff Nim (Wythoff 1907, Fraenkel 1982)

applying standard variations of such games

- move-size dynamics (Whinihan 1963, Schwenk 1970, Epp-Ferguson 1980, Zieve 1996, Holshouser-Reiter 2003, Gurvich 2012)
- Muller twists a.k.a Blocking maneuvers (Muller 1998, Holshouser-Reiter 2001, Smith-Stănică 2002)

2-pile Nim

Bouton's Nim is a 2-player game on a finite number of heaps, with given non-negative numbers of tokens. The players alternate in removing a positive number of tokens from precisely one of the heaps, at most a whole heap. A player who cannot move loses.

2-pile Nim

Bouton's Nim is a 2-player game on a finite number of heaps, with given non-negative numbers of tokens. The players alternate in removing a positive number of tokens from precisely one of the heaps, at most a whole heap. A player who cannot move loses.

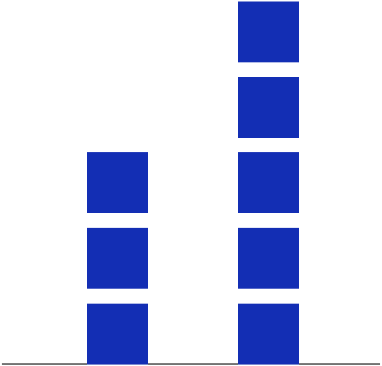
We play on two piles and let Alice be the first player, Bob the second.

2-pile Nim

2-pile Nim

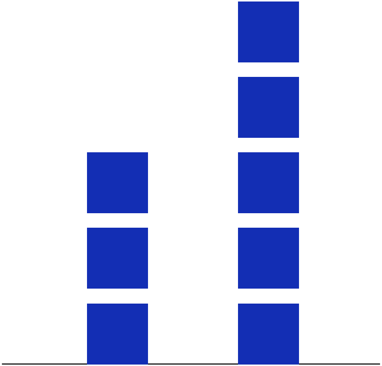
- Let the starting position be $(3, 5)$.

2-pile Nim



- Let the starting position be $(3, 5)$.

2-pile Nim

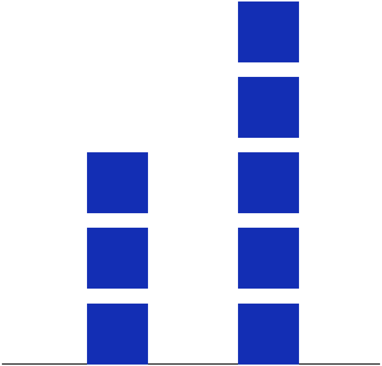


- Let the starting position be $(3, 5)$.
- Can Alice find a winning move?

2-pile Nim

$$\mathcal{P} = (0,0),(1,1),(2,2),\ldots$$

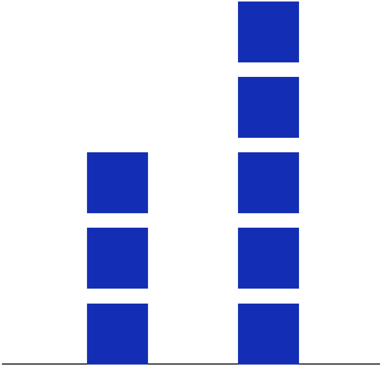
- Let the starting position be $(3,5)$.
- Can Alice find a winning move?



2-pile Nim

$$\mathcal{P} = (0,0), (1,1), (2,2), \ldots$$

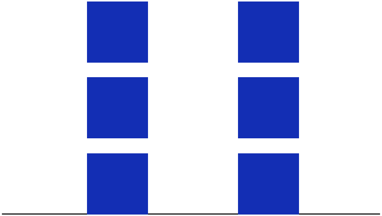
- Let the starting position be $(3, 5)$.
- Can Alice find a winning move?
- She removes two tokens from the larger pile.



2-pile Nim

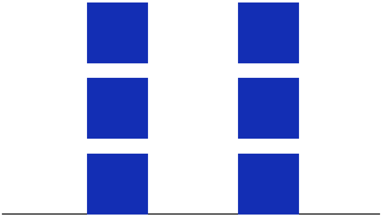
$$\mathcal{P} = (0,0), (1,1), (2,2), \ldots$$

- Let the starting position be $(3,5)$.
- Can Alice find a winning move?
- She removes two tokens from the larger pile.



2-pile Nim

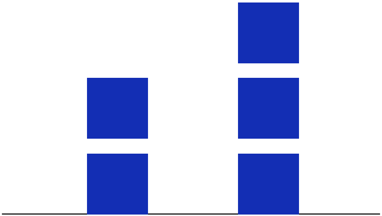
$$\mathcal{P} = (0,0), (1,1), (2,2), \ldots$$



- Let the starting position be $(3, 5)$.
- Can Alice find a winning move?
- She removes two tokens from the larger pile.
- Bob has to play so that the heaps get different sizes.

2-pile Nim

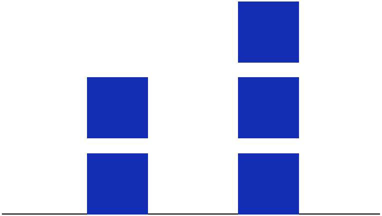
$$\mathcal{P} = (0,0), (1,1), (2,2), \ldots$$



- Let the starting position be $(3, 5)$.
- Can Alice find a winning move?
- She removes two tokens from the larger pile.
- Bob has to play so that the heaps get different sizes.

2-pile Nim

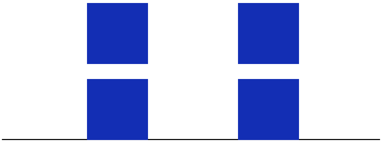
$$\mathcal{P} = (0,0),(1,1),(2,2),\ldots$$



- Suppose Alice decides to imitate Bob's move, by removing the same number of tokens, but from the other pile.

2-pile Nim

$$\mathcal{P} = (0,0), (1,1), (2,2), \ldots$$

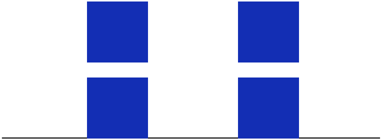


- Suppose Alice decides to imitate Bob's move, by removing the same number of tokens, but from the other pile.

2-pile Nim

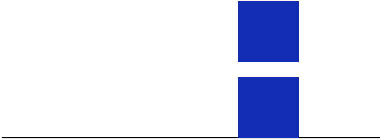
$$\mathcal{P} = (0,0), (1,1), (2,2), \ldots$$

- Suppose Alice decides to imitate Bob's move, by removing the same number of tokens, but from the other pile.
- Note that this will always be possible.



2-pile Nim

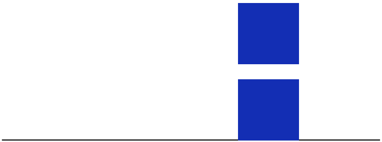
$$\mathcal{P} = (0,0), (1,1), (2,2), \ldots$$



- Suppose Alice decides to imitate Bob's move, by removing the same number of tokens, but from the other pile.
- Note that this will always be possible.

2-pile Nim

$$\mathcal{P} = (0,0),(1,1),(2,2),\ldots$$



- Suppose Alice decides to imitate Bob's move, by removing the same number of tokens, but from the other pile.
- Note that this will always be possible.
- The strategy is winning.

2-pile Nim

$$\mathcal{P} = (0,0),(1,1),(2,2),\ldots$$

- Suppose Alice decides to imitate Bob's move, by removing the same number of tokens, but from the other pile.
- Note that this will always be possible.
- The strategy is winning.

Our plan

In effect, Alice's winning strategy is to remove the same number of tokens from the larger heap as Bob removed from the smaller.

Our plan

In effect, Alice's winning strategy is to remove the same number of tokens from the larger heap as Bob removed from the smaller.

We will construct new games by restricting the number of such moves.

Our plan

In effect, Alice's winning strategy is to remove the same number of tokens from the larger heap as Bob removed from the smaller.

We will construct new games by restricting the number of such moves. For Nim, the number of such imitations is unlimited.

Our plan

In effect, Alice's winning strategy is to remove the same number of tokens from the larger heap as Bob removed from the smaller.

We will construct new games by restricting the number of such moves. For Nim, the number of such imitations is unlimited. What if we fix a number and say that repeated imitation beyond this number is not allowed?

Impartial games

Our games belong to the family of acyclic impartial games on a finite number of positions, where perfect play is possible. Two players alternate in moving; they follow the same game rules; there is no chance device, no hidden information and there is a final position which determines the winner of the game. We play the normal version where a player who cannot move loses.

Move-size dynamic imitations

Definition

Suppose that the heap sizes in a two heap take-away game are a and b and that the previous player removed $0 < x$ tokens from the a-heap, and where $a + x \leq b$. Then the next player imitates the previous player if he removes x tokens from the b-heap.

Move-size dynamic imitations

Definition

Suppose that the heap sizes in a two heap take-away game are a and b and that the previous player removed $0 < x$ tokens from the a-heap, and where $a + x \leq b$. Then the next player imitates the previous player if he removes x tokens from the b-heap.

Definition

In the impartial game of Imitation Nim the players move as in Nim, but imitations are not allowed. In k-Imitation Nim, at most $k - 1$ consecutive imitations by one and the same player are allowed.

The outcome of an impartial game

Definition

- a position is P if none of its options is P

The outcome of an impartial game

Definition

- a position is P if none of its options is P
- a position is N if there is a P-position in its set of options.

The outcome of an impartial game

Definition

- a position is P if none of its options is P
- a position is N if there is a P-position in its set of options.

The outcome of an impartial game

Definition

- a position is P if none of its options is P
- a position is N if there is a P-position in its set of options.

It follows: Each terminal position is P; the previous player will win if and only if the position is P.

W. A. Wythoff's Nim extension

The game of Wythoff Nim is an impartial game played on two piles of tokens. It was published 1907 in the article "A modification of the game of Nim" by W.A. Wythoff, a Dutch mathematician.

W. A. Wythoff's Nim extension

The game of Wythoff Nim is an impartial game played on two piles of tokens. It was published 1907 in the article "A modification of the game of Nim" by W.A. Wythoff, a Dutch mathematician. In addition to the rules of Nim, Wythoff allows removal of an equal number of tokens from each pile.

W. A. Wythoff's Nim extension

The game of Wythoff Nim is an impartial game played on two piles of tokens. It was published 1907 in the article "A modification of the game of Nim" by W.A. Wythoff, a Dutch mathematician. In addition to the rules of Nim, Wythoff allows removal of an equal number of tokens from each pile.

The game is maybe more known as the impartial game "Corner the Queen" (Rufus P. Isaacs, 1960), where the two players alternate in moving one single Queen-of-Chess, aiming to get her to the lower left corner of a (large) chessboard; taking into consideration that, by moving, the distance to this corner must decrease.

Adjoining P-positions as moves

Nim $\rightarrow$ Wythoff Nim, by adjoining all P-positions of Nim as moves.

Adjoining P-positions as moves

Nim $\rightarrow$ Wythoff Nim, by adjoining all P-positions of Nim as moves.
Indeed the P-positions of 2-pile Nim are $\{(k, k) \mid k \in \mathbb{Z}_{>0}\}$.

Adjoining P-positions as moves

Nim $\rightarrow$ Wythoff Nim, by adjoining all P-positions of Nim as moves. Indeed the P-positions of 2-pile Nim are $\{(k,k) \mid k \in \mathbb{Z}_{>0}\}$. Regarding move options as integers vectors to subtract from given positions, these are the “diagonal moves” of Corner-the-Queen.

The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W = \{(a_n, b_n), (b_n, a_n)\}$, can be computed recursively by a certain "minimal exclusive" rule:

$(0, 0)$,

The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W = \{(a_n, b_n), (b_n, a_n)\}$, can be computed recursively by a certain "minimal exclusive" rule:

$(0, 0), (1,$

The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W = \{(a_n, b_n), (b_n, a_n)\}$, can be computed recursively by a certain "minimal exclusive" rule:

$(0, 0), (1, 2),$

The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W = \{(a_n, b_n), (b_n, a_n)\}$, can be computed recursively by a certain "minimal exclusive" rule:

$(0, 0), (1, 2), (3,$

The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W = \{(a_n, b_n), (b_n, a_n)\}$, can be computed recursively by a certain "minimal exclusive" rule:

$(0, 0), (1, 2), (3, 5),$

The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W = \{(a_n, b_n), (b_n, a_n)\}$, can be computed recursively by a certain "minimal exclusive" rule:

$(0, 0), (1, 2), (3, 5), (4,$

The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W = \{(a_n, b_n), (b_n, a_n)\}$, can be computed recursively by a certain "minimal exclusive" rule:

$(0, 0), (1, 2), (3, 5), (4, 7)$, etc

The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W = \{(a_n, b_n), (b_n, a_n)\}$, can be computed recursively by a certain "minimal exclusive" rule:

$(0,0), (1,2), (3,5), (4,7)$, etc

Definition

For $X \subset \mathbb{Z}_{\geq 0}$ (strict subset), let $\text{mex}(X) := \min(\mathbb{Z}_{\geq 0} \setminus X)$.

The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W = \{(a_n, b_n), (b_n, a_n)\}$, can be computed recursively by a certain "minimal exclusive" rule:

$(0,0),(1,2),(3,5),(4,7)$, etc

Definition

For $X \subset \mathbb{Z}_{\geq 0}$ (strict subset), let $\mathrm{mex}(X) := \min(\mathbb{Z}_{\geq 0} \setminus X)$.

Theorem

For $n \geq 0$, $a_n = mex\{a_i, b_i \mid i < n\}, b_n = a_n + n$.

Complementary sequences

We say that two sequences of positive integers are complementary if each positive integer is contained in precisely one of them.

Complementary sequences

We say that two sequences of positive integers are complementary if each positive integer is contained in precisely one of them.

Theorem (Rayleigh 1894, Beatty 1926)

Suppose that $0 < \alpha < \beta$ *are positive real numbers such that* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. *Then* $(\lfloor i\alpha \rfloor)_{i=1}^{\infty}$ *and* $(\lfloor i\beta \rfloor)_{i=1}^{\infty}$ *are complementary if and only if* $1 < \alpha < 2 < \beta$ *are irrational.*

Complementary sequences

We say that two sequences of positive integers are complementary if each positive integer is contained in precisely one of them.

Theorem (Rayleigh 1894, Beatty 1926)

Suppose that $0 < \alpha < \beta$ *are positive real numbers such that* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. *Then* $(\lfloor i\alpha \rfloor)_{i=1}^{\infty}$ *and* $(\lfloor i\beta \rfloor)_{i=1}^{\infty}$ *are complementary if and only if* $1 < \alpha < 2 < \beta$ *are irrational.*

Beatty sequences

We call $(\lfloor i\alpha \rfloor)_{i=1}^{\infty}$ a Beatty sequence (of modulus α) if $\alpha > 0$ is irrational.

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ *denote the set of P-positions of Wythoff Nim, with* $a_i \leq b_i$*, for all* i*. Then*

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ *denote the set of P-positions of Wythoff Nim, with* $a_i \leq b_i$*, for all* i*. Then*

(i) (a_i) *and* (b_i) *are strictly increasing sequences;*

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ *denote the set of P-positions of Wythoff Nim, with* $a_i \leq b_i$*, for all* i*. Then*

(i) (a_i) *and* (b_i) *are strictly increasing sequences;*

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ *denote the set of P-positions of Wythoff Nim, with* $a_i \leq b_i$*, for all* i*. Then*

(i) (a_i) *and* (b_i) *are strictly increasing sequences;*

(ii) *they are complementary;*

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ denote the set of P-positions of Wythoff Nim, with $a_i \leq b_i$, for all i. Then

(i) *(a_i) and (b_i) are strictly increasing sequences;*

(ii) *they are complementary;*

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ denote the set of P-positions of Wythoff Nim, with $a_i \leq b_i$, for all i. Then

(i) *(a_i) and (b_i) are strictly increasing sequences;*

(ii) *they are complementary;*

(iii) *for all i, $b_i - a_i = i$ and*

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ *denote the set of P-positions of Wythoff Nim, with* $a_i \leq b_i$*, for all* i*. Then*

(i) (a_i) *and* (b_i) *are strictly increasing sequences;*

(ii) *they are complementary;*

(iii) *for all* i*,* $b_i - a_i = i$ *and*

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ *denote the set of P-positions of Wythoff Nim, with* $a_i \leq b_i$*, for all* i*. Then*

(i) (a_i) *and* (b_i) *are strictly increasing sequences;*

(ii) *they are complementary;*

(iii) *for all* i, $b_i - a_i = i$ *and*

(iv) $(a_i, b_i) = (\lfloor \phi i \rfloor, \lfloor \phi^2 i \rfloor)$, *where* $\phi = \frac{1+\sqrt{5}}{2}$, *the golden ratio.*

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ *denote the set of P-positions of Wythoff Nim, with* $a_i \leq b_i$*, for all* i*. Then*

(i) (a_i) *and* (b_i) *are strictly increasing sequences;*

(ii) *they are complementary;*

(iii) *for all* i, $b_i - a_i = i$ *and*

(iv) $(a_i, b_i) = (\lfloor \phi i \rfloor, \lfloor \phi^2 i \rfloor)$*, where* $\phi = \frac{1+\sqrt{5}}{2}$*, the golden ratio.*

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ denote the set of P-positions of Wythoff Nim, with $a_i \leq b_i$, for all i. Then

(i) *(a_i) and (b_i) are strictly increasing sequences;*

(ii) *they are complementary;*

(iii) *for all i, $b_i - a_i = i$ and*

(iv) *$(a_i, b_i) = (\lfloor \phi i \rfloor, \lfloor \phi^2 i \rfloor)$, where $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio.*

Theorem (Larsson, 2009)

The P-positions of Wythoff Nim are identical to those of Imitation Nim, regarded as starting position.

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

Let $\mathcal{P}_W = \{(a_i, b_i), (b_i, a_i)\}_{i=0}^{\infty}$ denote the set of P-positions of Wythoff Nim, with $a_i \leq b_i$, for all i. Then

(i) *(a_i) and (b_i) are strictly increasing sequences;*

(ii) *they are complementary;*

(iii) *for all i, $b_i - a_i = i$ and*

(iv) *$(a_i, b_i) = (\lfloor \phi i \rfloor, \lfloor \phi^2 i \rfloor)$, where $\phi = \frac{1+\sqrt{5}}{2}$, the golden ratio.*

Theorem (Larsson, 2009)

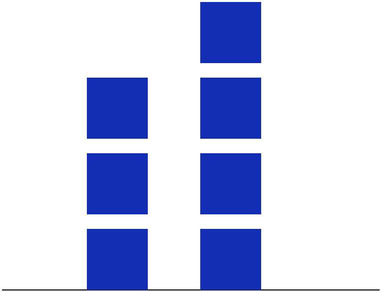
The P-positions of Wythoff Nim are identical to those of Imitation Nim, regarded as starting position.

The proof uses a simple inductive argument. We let Alice and Bob illustrate the idea.

$\mathcal{P}_W = (0,0),(1,2),(3,5),\ldots$

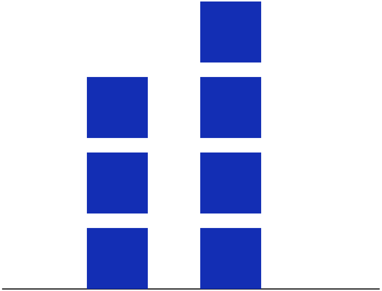
- Suppose our starting position is an *N*-position of Wythoff Nim, say $(3,4)$.

$\mathcal{P}_W =$
$(0,0),(1,2),(3,5),\ldots$



- Suppose our starting position is an *N*-position of Wythoff Nim, say $(3,4)$.

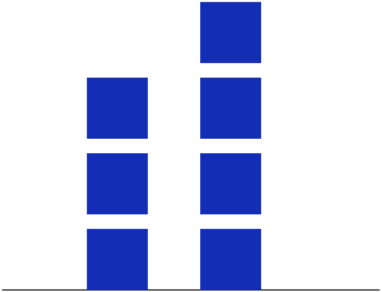
$\mathcal{P}_W =$
$(0,0),(1,2),(3,5),\ldots$



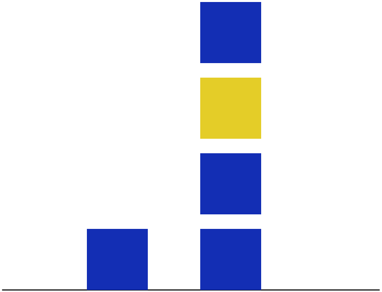
- Suppose our starting position is an N-position of Wythoff Nim, say $(3,4)$.
- We want to show that this position is an N-position of Imitation Nim.

$\mathcal{P}_W =$
$(0,0),(1,2),(3,5),\ldots$

- Suppose our starting position is an N-position of Wythoff Nim, say $(3,4)$.
- We want to show that this position is an N-position of Imitation Nim.
- Alice may invoke the imitation rule, by removing two tokens from the smaller pile.



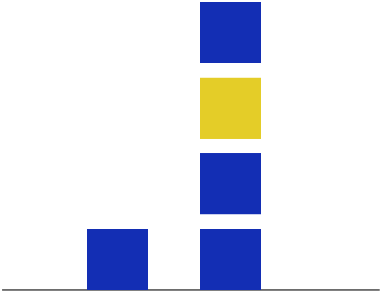
$\mathcal{P}_W =$
$(0,0),(1,2),(3,5),\ldots$



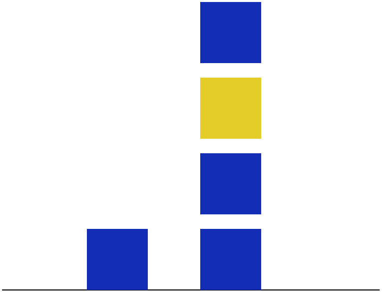
- Suppose our starting position is an N-position of Wythoff Nim, say $(3,4)$.
- We want to show that this position is an N-position of Imitation Nim.
- Alice may invoke the imitation rule, by removing two tokens from the smaller pile.

$\mathcal{P}_W =$
$(0,0),(1,2),(3,5),\ldots$

- Suppose our starting position is an N-position of Wythoff Nim, say $(3,4)$.
- We want to show that this position is an N-position of Imitation Nim.
- Alice may invoke the imitation rule, by removing two tokens from the smaller pile.
- Therefore Bob cannot move to (the P-position of Wythoff Nim) $(1,2)$.



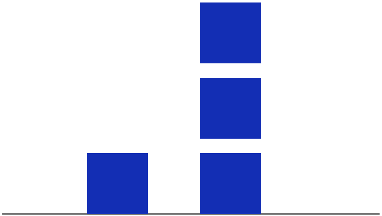
$\mathcal{P}_W =$
$(0,0),(1,2),(3,5),\ldots$



- Suppose our starting position is an *N*-position of Wythoff Nim, say $(3,4)$.
- We want to show that this position is an *N*-position of Imitation Nim.
- Alice may invoke the imitation rule, by removing two tokens from the smaller pile.
- Therefore Bob cannot move to (the *P*-position of Wythoff Nim) $(1,2)$.
- There are no other P-positions of Wythoff Nim as options. Induction gives the claim.

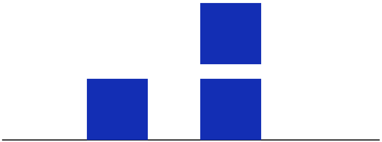
$\mathcal{P}_W = (0,0),(1,2),(3,5),\ldots$

- Suppose our starting position is an *N*-position of Wythoff Nim, say $(3,4)$.
- We want to show that this position is an *N*-position of Imitation Nim.
- Alice may invoke the imitation rule, by removing two tokens from the smaller pile.
- Therefore Bob cannot move to (the *P*-position of Wythoff Nim) $(1,2)$.
- There are no other P-positions of Wythoff Nim as options. Induction gives the claim.
- ...Bob moves to $(1,3)$



$\mathcal{P}_W =$
$(0,0),(1,2),(3,5),\ldots$

- Suppose our starting position is an N-position of Wythoff Nim, say $(3,4)$.
- We want to show that this position is an N-position of Imitation Nim.
- Alice may invoke the imitation rule, by removing two tokens from the smaller pile.
- Therefore Bob cannot move to (the P-position of Wythoff Nim) $(1,2)$.
- There are no other P-positions of Wythoff Nim as options. Induction gives the claim.
- ...Bob moves to $(1,3)$
- but Alice responds by moving to $(1,2)$ and wins.



2-Imitation Nim

Example:

Let us play 2-Imitation Nim: at most one imitation is allowed. Let the starting position be $(2, 2)$.

2-Imitation Nim

Example:

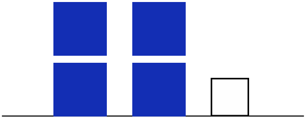
Let us play 2-Imitation Nim: at most one imitation is allowed. Let the starting position be $(2, 2)$.

Notation:

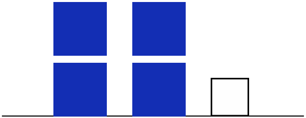
The default color of a token is blue. A token is green if removal of it implies that an *imitation counter* is increased by one. A token is yellow if it may not be removed. The imitation counter is drawn as a black square.

Example: At most one imitation

- The starting position is $(2, 2)$.



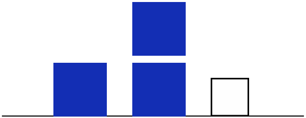
Example: At most one imitation



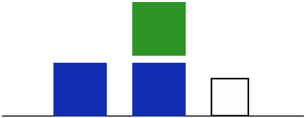
- The starting position is $(2, 2)$.
- Alice moves to $(1, 2)$;

Example: At most one imitation

- The starting position is $(2, 2)$.
- Alice moves to $(1, 2)$;



Example: At most one imitation



- The starting position is $(2, 2)$.
- Alice moves to $(1, 2)$;
- Removing a green token means to imitate the previous move;

Example: At most one imitation

- The starting position is $(2, 2)$.
- Alice moves to $(1, 2)$;
- Removing a green token means to imitate the previous move;
- If Bob moves to $(0, 2)$ he will lose right away, so he rather moves to $(1, 1)$;

Example: At most one imitation

- The starting position is $(2, 2)$.
- Alice moves to $(1, 2)$;
- Removing a green token means to imitate the previous move;
- If Bob moves to $(0, 2)$ he will lose right away, so he rather moves to $(1, 1)$;
- The imitation counter increases by one unit;

Example: At most one imitation

- The starting position is $(2, 2)$.
- Alice moves to $(1, 2)$;
- Removing a green token means to imitate the previous move;
- If Bob moves to $(0, 2)$ he will lose right away, so he rather moves to $(1, 1)$;
- The imitation counter increases by one unit;
- Alice moves to $(0, 1)$;

Example: At most one imitation

- The starting position is $(2, 2)$.
- Alice moves to $(1, 2)$;
- Removing a green token means to imitate the previous move;
- If Bob moves to $(0, 2)$ he will lose right away, so he rather moves to $(1, 1)$;
- The imitation counter increases by one unit;
- Alice moves to $(0, 1)$;
- This time, Bob cannot imitate, in fact he can't move at all.

Example: At most one imitation

- The starting position is $(2, 2)$.
- Alice moves to $(1, 2)$;
- Removing a green token means to imitate the previous move;
- If Bob moves to $(0, 2)$ he will lose right away, so he rather moves to $(1, 1)$;
- The imitation counter increases by one unit;
- Alice moves to $(0, 1)$;
- This time, Bob cannot imitate, in fact he can't move at all.
- So $(2, 2)$ is an N-position for 2-Imitation Nim, unlike 2-pile Nim.

A Muller twist on Wythoff's game

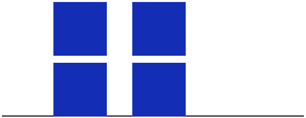
Our next game is 2-Blocking Wythoff Nim. At each stage of game, the previous player may, before the next player moves, "block off" at most one diagonal option from the set of Wythoff Nim options. Any blocking maneuver is forgotten immediately after the move is carried out.

A Muller twist on Wythoff's game

Our next game is 2-Blocking Wythoff Nim. At each stage of game, the previous player may, before the next player moves, "block off" at most one diagonal option from the set of Wythoff Nim options. Any blocking maneuver is forgotten immediately after the move is carried out.

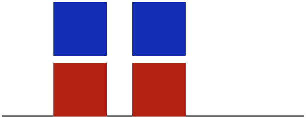
A pair of tokens is painted red if it (together with the tokens on top), may not be removed. Notice that one of the red tokens may be removed, but not both at the same time.

Example: One diagonal option may be blocked



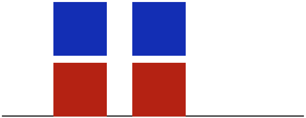
- The starting position is $(2, 2)$.

Example: One diagonal option may be blocked



- The starting position is $(2, 2)$.
- Bob blocks off removal of all tokens;

Example: One diagonal option may be blocked



- The starting position is $(2, 2)$.
- Bob blocks off removal of all tokens;
- But Bob can only prolong Alice's path to victory;

Example: One diagonal option may be blocked

- The starting position is $(2, 2)$.
- Bob blocks off removal of all tokens;
- But Bob can only prolong Alice's path to victory;
- Alice moves to $(1, 1)$ and makes the obvious block;

Example: One diagonal option may be blocked

- The starting position is $(2, 2)$.
- Bob blocks off removal of all tokens;
- But Bob can only prolong Alice's path to victory;
- Alice moves to $(1, 1)$ and makes the obvious block;

Example: One diagonal option may be blocked

- The starting position is $(2, 2)$.
- Bob blocks off removal of all tokens;
- But Bob can only prolong Alice's path to victory;
- Alice moves to $(1, 1)$ and makes the obvious block;
- Bob may remove either one of the tokens, but not both;

Example: One diagonal option may be blocked

- The starting position is $(2, 2)$.
- Bob blocks off removal of all tokens;
- But Bob can only prolong Alice's path to victory;
- Alice moves to $(1, 1)$ and makes the obvious block;
- Bob may remove either one of the tokens, but not both;
- and since a single token cannot be blocked off,

Example: One diagonal option may be blocked

- The starting position is $(2, 2)$.
- Bob blocks off removal of all tokens;
- But Bob can only prolong Alice's path to victory;
- Alice moves to $(1, 1)$ and makes the obvious block;
- Bob may remove either one of the tokens, but not both;
- and since a single token cannot be blocked off,
- Alice wins, so $(2, 2)$ is a next player winning position...

Example: One diagonal option may be blocked

- The starting position is $(2, 2)$.
- Bob blocks off removal of all tokens;
- But Bob can only prolong Alice's path to victory;
- Alice moves to $(1, 1)$ and makes the obvious block;
- Bob may remove either one of the tokens, but not both;
- and since a single token cannot be blocked off,
- Alice wins, so $(2, 2)$ is a next player winning position...
- ...just as for 2-Imitation Nim.

The generalized algorithm

For fixed positive integers k and m and all $n \geq 0$, let

$$a_n = \text{mex}\{a_i, b_i \mid i < n\}, b_n = a_n + \left\lfloor \frac{m}{k} n \right\rfloor .$$

The generalized algorithm

For fixed positive integers k and m and all $n \geq 0$, let

$$a_n = \text{mex}\{a_i, b_i \mid i < n\}, b_n = a_n + \left\lfloor \frac{m}{k} n \right\rfloor .$$

Clearly (a_n) and (b_n) are complementary. Questions:

- Is $\{(a_n, b_n), (b_n, a_n)\}$ the set of P-position of some take-away game?

The generalized algorithm

For fixed positive integers k and m and all $n \geq 0$, let

$$a_n = \text{mex}\{a_i, b_i \mid i < n\}, b_n = a_n + \left\lfloor \frac{m}{k} n \right\rfloor.$$

Clearly (a_n) and (b_n) are complementary. Questions:

- Is $\{(a_n, b_n), (b_n, a_n)\}$ the set of P-position of some take-away game?
- Can (a_n) and (b_n) be described via Beatty sequences?

The generalized algorithm

For fixed positive integers k and m and all $n \geq 0$, let

$$a_n = \text{mex}\{a_i, b_i \mid i < n\}, b_n = a_n + \left\lfloor \frac{m}{k} n \right\rfloor .$$

Clearly (a_n) and (b_n) are complementary. Questions:

- Is $\{(a_n, b_n), (b_n, a_n)\}$ the set of P-position of some take-away game?
- Can (a_n) and (b_n) be described via Beatty sequences?
- If not, is the decision problem wether (x, y) is of the form (a_n, b_n) for some n tractable (polynomial in succinct input size)?

Fraenkel's game with a diagonal Muller twist

Definition

The game of (k, m)-Blocking Wythoff Nim, $W_{k,m}$, is the game where the next player

I. moves as in 2-pile Nim, or

Fraenkel's game with a diagonal Muller twist

Definition

The game of (k, m)-Blocking Wythoff Nim, $W_{k,m}$, is the game where the next player

I. moves as in 2-pile Nim, or

II. removes $0 < i$ tokens from one of the piles and remove $0 < j$ tokens from the other pile as long as $0 \leq | j - i | < m$ (Fraenkel's m-Wythoff Nim 1982)

Fraenkel's game with a diagonal Muller twist

Definition

The game of (k, m)-Blocking Wythoff Nim, $W_{k,m}$, is the game where the next player

I. moves as in 2-pile Nim, or

II. removes $0 < i$ tokens from one of the piles and remove $0 < j$ tokens from the other pile as long as $0 \leq | j - i | < m$ (Fraenkel's m-Wythoff Nim 1982)

III. but before the next player makes his move, the previous player may declare at most $k - 1$ of the diagonal options, i.e. with $i = j$, as blocked (Hegarty, Larsson 2006).

Arithmetic properties of $W_{k,m}$

Proposition (Hegarty, Larsson 2006, 2009)

(i) *The set of P-positions of (k, m)-Blocking Wythoff Nim is $\{(a_i, b_i), (b_i, a_i)\}$ (generalized form);*

Arithmetic properties of $W_{k,m}$

Proposition (Hegarty, Larsson 2006, 2009)

(i) *The set of P-positions of* (k, m)*-Blocking Wythoff Nim is* $\{(a_i, b_i), (b_i, a_i)\}$ *(generalized form);*

(ii) (a_n) *and* (b_n) *are strictly increasing;*

Arithmetic properties of $W_{k,m}$

Proposition (Hegarty, Larsson 2006, 2009)

(i) *The set of P-positions of (k, m)-Blocking Wythoff Nim is* $\{(a_i, b_i), (b_i, a_i)\}$ *(generalized form);*

(ii) (a_n) *and* (b_n) *are strictly increasing;*

(iii) $\#\{i \in \mathbf{N} \mid b_i - a_i = d\} = k$ *if* $m \mid d$, *otherwise* $\#\{i \in \mathbf{N} \mid b_i - a_i = d\} = 0$;

Arithmetic properties of $W_{k,m}$

Proposition (Hegarty, Larsson 2006, 2009)

(i) *The set of P-positions of* (k, m)*-Blocking Wythoff Nim is* $\{(a_i, b_i), (b_i, a_i)\}$ *(generalized form);*

(ii) (a_n) *and* (b_n) *are strictly increasing;*

(iii) $\#\{i \in \mathbf{N} \mid b_i - a_i = d\} = k$ *if* $m \mid d$*, otherwise* $\#\{i \in \mathbf{N} \mid b_i - a_i = d\} = 0$*;*

(iv) *Suppose* (a, b) *and* (c, d) *are two distinct P-positions of a game of* (k, m)*-Blocking Wythoff Nim. Then* $a < c$ *implies* $b - a \leq d - c$ *(and* $b < d$*).*

Arithmetic properties of $W_{k,m}$

Proposition (Hegarty, Larsson 2006, 2009)

(i) *The set of P-positions of (k, m)-Blocking Wythoff Nim is $\{(a_i, b_i), (b_i, a_i)\}$ (generalized form);*

(ii) *(a_n) and (b_n) are strictly increasing;*

(iii) *$\#\{i \in \mathbf{N} \mid b_i - a_i = d\} = k$ if $m \mid d$, otherwise $\#\{i \in \mathbf{N} \mid b_i - a_i = d\} = 0$;*

(iv) *Suppose (a, b) and (c, d) are two distinct P-positions of a game of (k, m)-Blocking Wythoff Nim. Then $a < c$ implies $b - a \leq d - c$ (and $b < d$).*

(v) *Suppose $b_i \geq a_i$. Then $\lim_{i\to\infty} \frac{b_i}{a_i}$ exists and is given by the positive root of $x^2 - \frac{m}{k}x - 1 = 0$, but*

Arithmetic properties of $W_{k,m}$

Proposition (Hegarty, Larsson 2006, 2009)

(i) *The set of P-positions of (k, m)-Blocking Wythoff Nim is* $\{(a_i, b_i), (b_i, a_i)\}$ *(generalized form);*

(ii) (a_n) *and* (b_n) *are strictly increasing;*

(iii) $\#\{i \in \mathbf{N} \mid b_i - a_i = d\} = k$ *if* $m \mid d$*, otherwise* $\#\{i \in \mathbf{N} \mid b_i - a_i = d\} = 0$*;*

(iv) *Suppose* (a, b) *and* (c, d) *are two distinct P-positions of a game of* (k, m)*-Blocking Wythoff Nim. Then* $a < c$ *implies* $b - a \leq d - c$ *(and* $b < d$*).*

(v) *Suppose* $b_i \geq a_i$*. Then* $\lim_{i\to\infty} \frac{b_i}{a_i}$ *exists and is given by the positive root of* $x^2 - \frac{m}{k}x - 1 = 0$*, but*

(vi) (a_i) *and* (b_i) *can be expressed by Beatty sequences only for special cases, namely if* $k \mid m$*. (Note: if* $k = 1$ *we have Fraenkel's m-Wythoff Nim.)*

A polynomial time approach

In the Appendix of my paper '2-Pile Nim with a restricted number of move-size imitations', P. Hegarty shows that if $k > 1$ and $m = 1$, the sequences are "close to" Beatty sequences with

$$\alpha = \frac{2k - m + \sqrt{m^2 + 4k^2}}{2k}, \beta = \alpha + \frac{m}{k}.$$

A polynomial time approach

In the Appendix of my paper '2-Pile Nim with a restricted number of move-size imitations', P. Hegarty shows that if $k > 1$ and $m = 1$, the sequences are "close to" Beatty sequences with

$$\alpha = \frac{2k - m + \sqrt{m^2 + 4k^2}}{2k}, \beta = \alpha + \frac{m}{k}.$$

Via the general bound, for all n,

$$\lfloor (n - k - 1)\alpha \rfloor \leq a_n \leq \lfloor n\alpha \rfloor,$$

a polynomial time algorithm to determine $\mathcal{P}_{W_{k,m}}$ is developed by Udi Peled in his master thesis "Polynomializing seemingly hard sequences using surrogate sequences" (advisor A. S. Fraenkel) + paper. This has been generalized further by V. Gurvich (2012), to non-complementary sequences generated by a generalized mex-function.

A dynamic counting of P-positions of $W_{k,m}$

Definition

Let (a, b) be a position of $W_{k,m}$. Then

$$\xi((a, b)) := \#\{(i, j) \in \mathcal{P}_{W_{k,m}} \mid i < a, j - i = b - a\}.$$

A dynamic counting of P-positions of $W_{k,m}$

Definition

Let (a, b) be a position of $W_{k,m}$. Then

$$\xi((a,b)) := \#\{(i,j) \in \mathcal{P}_{W_{k,m}} \mid i < a, j - i = b - a\}.$$

Notice that at each stage of game, for a previous player winning strategy, at least $\xi((a,b))$ positions must be blocked off.

Imitations and how to count them

Definition

Let m be a positive integer. Suppose that the previous player removed x tokens from a smaller (or equal) pile. Then if the next player removes $x + i$ tokens from the other pile, where $0 \leq i < m$, he *m*-imitates (or imitates) the previous player's move.

Imitations and how to count them

Definition

Let m be a positive integer. Suppose that the previous player removed x tokens from a smaller (or equal) pile. Then if the next player removes $x + i$ tokens from the other pile, where $0 \leq i < m$, he *m-imitates* (or imitates) the previous player's move.

Definition

For a fixed m, suppose the current position of a 2-pile take-away game is X and the last two moves are $Z \to Y \to X$. Then put

$$L(X) := L(Z) - 1$$

if $Y \to X$ m-imitates $Z \to Y$. Otherwise put $L(X) := k - 1$.

Imitations and how to count them

Definition

Let m be a positive integer. Suppose that the previous player removed x tokens from a smaller (or equal) pile. Then if the next player removes $x + i$ tokens from the other pile, where $0 \leq i < m$, he m-imitates (or imitates) the previous player's move.

Definition

For a fixed m, suppose the current position of a 2-pile take-away game is X and the last two moves are $Z \to Y \to X$. Then put

$$L(X) := L(Z) - 1$$

if $Y \to X$ m-imitates $Z \to Y$. Otherwise put $L(X) := k - 1$.

In particular, $L(X) = k - 1$ if X is a starting position.

The game of (k, m)-Imitation Nim

Definition

Let $k, m \in \mathbb{Z}_{>0}$. The game of (k, m)-Imitation Nim (or Imitation Nim) is a take-away game on two piles of tokens, where the players

The game of (k, m)-Imitation Nim

Definition

Let $k, m \in \mathbb{Z}_{>0}$. The game of (k, m)-Imitation Nim (or Imitation Nim) is a take-away game on two piles of tokens, where the players

- move as in 2-pile Nim, but

The game of (k, m)-Imitation Nim

Definition

Let $k, m \in \mathbb{Z}_{>0}$. The game of (k, m)-Imitation Nim (or Imitation Nim) is a take-away game on two piles of tokens, where the players

- move as in 2-pile Nim, but
- with the restriction that no more than $k - 1$ consecutive m-imitations by one and the same player are permitted;

The game of (k, m)-Imitation Nim

Definition

Let $k, m \in \mathbb{Z}_{>0}$. The game of (k, m)-Imitation Nim (or Imitation Nim) is a take-away game on two piles of tokens, where the players

- move as in 2-pile Nim, but
- with the restriction that no more than $k - 1$ consecutive m-imitations by one and the same player are permitted;
- that is, a player may not move to a Nim-option, say X, if it would imply $L(X) = -1$.

The game of (k, m)-Imitation Nim

Definition

Let $k, m \in \mathbb{Z}_{>0}$. The game of (k, m)-Imitation Nim (or Imitation Nim) is a take-away game on two piles of tokens, where the players

- move as in 2-pile Nim, but
- with the restriction that no more than $k - 1$ consecutive m-imitations by one and the same player are permitted;
- that is, a player may not move to a Nim-option, say X, if it would imply $L(X) = -1$.

Note: At the Integers 2007 conference, A. Fraenkel suggested the name "Limitation Nim" for the case $k = 1$.

Comparing the number of options for modified games

Let $k, m \in \mathbb{Z}_{>0}$. How does the number of options of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim vary as we alter k and m?

Comparing the number of options for modified games

Let $k, m \in \mathbb{Z}_{>0}$. How does the number of options of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim vary as we alter k and m? Suppose we start with 2-pile Nim, then the modified game has:

Comparing the number of options for modified games

Let $k, m \in \mathbb{Z}_{>0}$. How does the number of options of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim vary as we alter k and m? Suppose we start with 2-pile Nim, then the modified game has:

- on the one hand, for the "Wythoff Nim-setting", if we
- on the other hand, for "Imitation Nim-setting", if we

Comparing the number of options for modified games

Let $k, m \in \mathbb{Z}_{>0}$. How does the number of options of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim vary as we alter k and m? Suppose we start with 2-pile Nim, then the modified game has:

- on the one hand, for the "Wythoff Nim-setting", if we
  - adjoin P-positions as moves—more options;
- on the other hand, for "Imitation Nim-setting", if we

Comparing the number of options for modified games

Let $k, m \in \mathbb{Z}_{>0}$. How does the number of options of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim vary as we alter k and m? Suppose we start with 2-pile Nim, then the modified game has:

- on the one hand, for the "Wythoff Nim-setting", if we
  - adjoin P-positions as moves—more options;
- on the other hand, for "Imitation Nim-setting", if we
  - limit the number of imitations—less options;

Comparing the number of options for modified games

Let $k, m \in \mathbb{Z}_{>0}$. How does the number of options of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim vary as we alter k and m? Suppose we start with 2-pile Nim, then the modified game has:

- on the one hand, for the "Wythoff Nim-setting", if we
  - adjoin P-positions as moves—more options;
  - increase k, more options may be blocked—less options;
- on the other hand, for "Imitation Nim-setting", if we
  - limit the number of imitations—less options;

Comparing the number of options for modified games

Let $k, m \in \mathbb{Z}_{>0}$. How does the number of options of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim vary as we alter k and m? Suppose we start with 2-pile Nim, then the modified game has:

- on the one hand, for the "Wythoff Nim-setting", if we
  - adjoin P-positions as moves—more options;
  - increase k, more options may be blocked—less options;
- on the other hand, for "Imitation Nim-setting", if we
  - limit the number of imitations—less options;
  - increase k, more imitations are allowed—more options;

Comparing the number of options for modified games

Let $k, m \in \mathbb{Z}_{>0}$. How does the number of options of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim vary as we alter k and m? Suppose we start with 2-pile Nim, then the modified game has:

- on the one hand, for the "Wythoff Nim-setting", if we
  - adjoin P-positions as moves—more options;
  - increase k, more options may be blocked—less options;
  - increase m, diagonal moves widens—more options.
- on the other hand, for "Imitation Nim-setting", if we
  - limit the number of imitations—less options;
  - increase k, more imitations are allowed—more options;

Comparing the number of options for modified games

Let $k, m \in \mathbb{Z}_{>0}$. How does the number of options of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim vary as we alter k and m? Suppose we start with 2-pile Nim, then the modified game has:

- on the one hand, for the "Wythoff Nim-setting", if we
  - adjoin P-positions as moves—more options;
  - increase k, more options may be blocked—less options;
  - increase m, diagonal moves widens—more options.
- on the other hand, for "Imitation Nim-setting", if we
  - limit the number of imitations—less options;
  - increase k, more imitations are allowed—more options;
  - increase m, more moves are imitations— less options.

Main Theorem

Theorem (Larsson 2009)

Let $k, m \in \mathbb{Z}_{>0}$.

- *Then* (x, y) *is a P-position of* (k, m)*-Imitation Nim if it is a P-position of* (k, m)*-Blocking Wythoff Nim and* $k - 1 \geq L(x, y) \geq \xi(x, y) \geq 0$. *In particular this holds for* (x, y) *a starting position of Imitation Nim.*

Main Theorem

Theorem (Larsson 2009)

Let $k, m \in \mathbb{Z}_{>0}$.

- *Then (x, y) is a P-position of (k, m)-Imitation Nim if it is a P-position of (k, m)-Blocking Wythoff Nim and $k - 1 \geq L(x, y) \geq \xi(x, y) \geq 0$. In particular this holds for (x, y) a starting position of Imitation Nim.*
- *The "dynamic" P-positions of (k, m)-Imitation Nim are of the form $(x, y + z)$ where (x, y), $x \leq y$, is an upper P-position of (k, m)-Blocking Wythoff Nim and where $-1 \leq L(x, y) < \xi(x, y) \leq k - 1$. (Similar for "lower" positions.)*

Main Theorem

Theorem (Larsson 2009)

Let $k, m \in \mathbb{Z}_{>0}$.

- *Then* (x, y) *is a P-position of* (k, m)*-Imitation Nim if it is a P-position of* (k, m)*-Blocking Wythoff Nim and* $k - 1 \geq L(x, y) \geq \xi(x, y) \geq 0$. *In particular this holds for* (x, y) *a starting position of Imitation Nim.*
- *The "dynamic" P-positions of* (k, m)*-Imitation Nim are of the form* $(x, y + z)$ *where* (x, y), $x \leq y$, *is an upper P-position of* (k, m)*-Blocking Wythoff Nim and where* $-1 \leq L(x, y) < \xi(x, y) \leq k - 1$. *(Similar for "lower" positions.)*
- *Any other position of Imitation Nim is an N-position.*

Summary

Summary

- The game of 2-pile Nim may be viewed as an imitation game, where a player may imitate the previous player's moves arbitrarily many times. The next player imitates the previous player's move if he removes the same number of tokens from a larger pile as the previous player removed from a smaller (or equal) pile.

Summary

- The game of 2-pile Nim may be viewed as an imitation game, where a player may imitate the previous player's moves arbitrarily many times. The next player imitates the previous player's move if he removes the same number of tokens from a larger pile as the previous player removed from a smaller (or equal) pile.

Summary

- The game of 2-pile Nim may be viewed as an imitation game, where a player may imitate the previous player's moves arbitrarily many times. The next player imitates the previous player's move if he removes the same number of tokens from a larger pile as the previous player removed from a smaller (or equal) pile.
- Wythoff Nim can be viewed as the game where we to 2-pile Nim adjoin the P-positions as options.

Summary

- The game of 2-pile Nim may be viewed as an imitation game, where a player may imitate the previous player's moves arbitrarily many times. The next player imitates the previous player's move if he removes the same number of tokens from a larger pile as the previous player removed from a smaller (or equal) pile.
- Wythoff Nim can be viewed as the game where we to 2-pile Nim adjoin the P-positions as options.

Summary

- The game of 2-pile Nim may be viewed as an imitation game, where a player may imitate the previous player's moves arbitrarily many times. The next player imitates the previous player's move if he removes the same number of tokens from a larger pile as the previous player removed from a smaller (or equal) pile.
- Wythoff Nim can be viewed as the game where we to 2-pile Nim adjoin the P-positions as options.
- Limitation Nim $=$ $(1, 1)$-Imitation Nim is the game where the next player may not imitate the previous player's most recent move. This game has the same P-positions as Wythoff Nim, if one only regards the starting positions.

Summary

- If we put a Muller twist to a game of Wythoff Nim, where we allow the previous player to block off at most $k - 1 \geq 0$ of the next player's diagonal options, then, regarded as starting positions, we get identical P-positions as for k-Imitation Nim. For the latter game, at most $k - 1$ consecutive imitations from one and the same player is permitted.

Summary

- If we put a Muller twist to a game of Wythoff Nim, where we allow the previous player to block off at most $k-1 \geq 0$ of the next player's diagonal options, then, regarded as starting positions, we get identical P-positions as for k-Imitation Nim. For the latter game, at most $k-1$ consecutive imitations from one and the same player is permitted.

Summary

- If we put a Muller twist to a game of Wythoff Nim, where we allow the previous player to block off at most $k - 1 \geq 0$ of the next player's diagonal options, then, regarded as starting positions, we get identical P-positions as for k-Imitation Nim. For the latter game, at most $k - 1$ consecutive imitations from one and the same player is permitted.
- A "wider m-diagonal" of Wythoff Nim corresponds to an "m-relaxed" notion of an imitation. What is more, there is a precise dynamic correspondence between the winning positions of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim. This relationship constitutes our main theorem.

Question

- Do limited imitations and Muller twists also have some interesting interpretations for classical (bimatrix) games?