Imitation Nim

Urban Larsson, Göteborg, Sweden

The Stony Brook (satellite) workshop

July 18, 2013

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- move-size dynamics (Whinihan 1963, Schwenk 1970, Epp-Ferguson 1980, Zieve 1996, Holshouser-Reiter 2003, Gurvich 2012)
- Muller twists a.k.a Blocking maneuvers (Muller 1998, Holshouser-Reiter 2001, Smith-Stănică 2002)

2-pile Nim

Bouton's Nim is a 2-player game on a finite number of heaps, with given non-negative numbers of tokens. The players alternate in removing a positive number of tokens from precisely one of the heaps, at most a whole heap. A player who cannot move loses.

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We play on two piles and let Alice be the first player, Bob the second.

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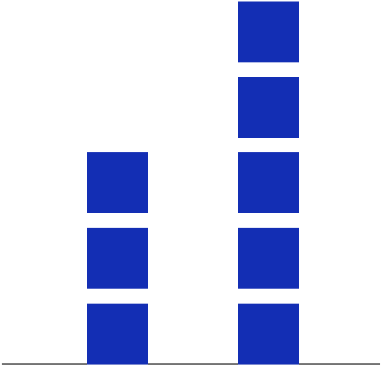

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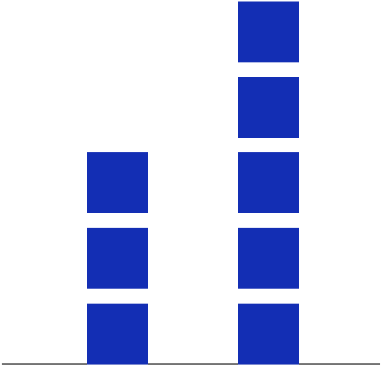

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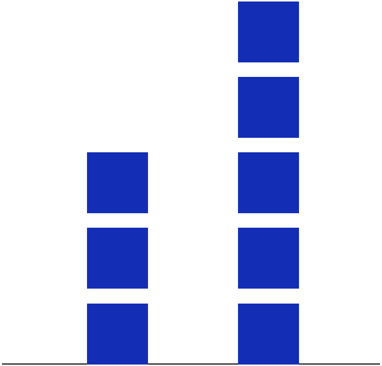

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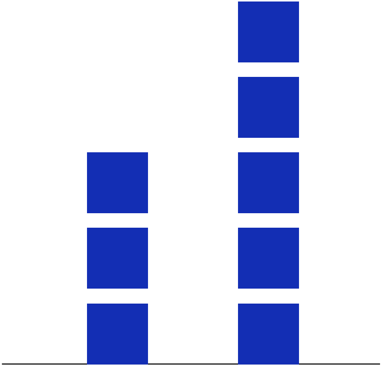

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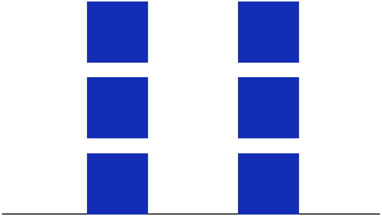

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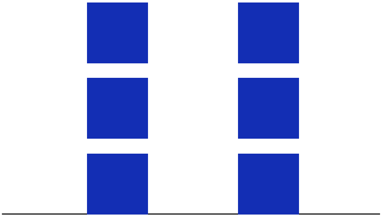

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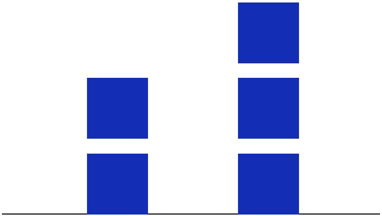

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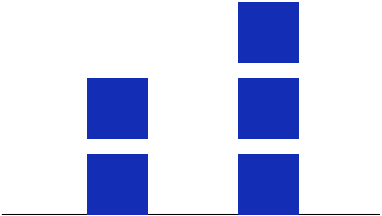

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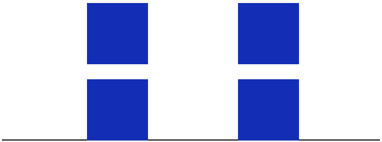

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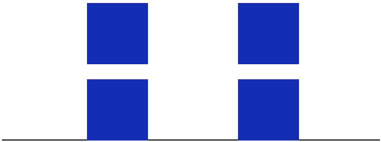

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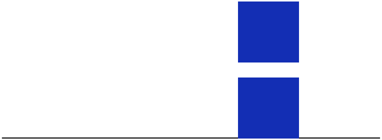

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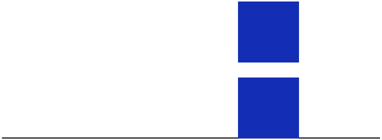

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We will construct new games by restricting the number of such moves. For Nim, the number of such imitations is unlimited. What if we fix a number and say that repeated imitation beyond this number is not allowed?

Impartial games

Our games belong to the family of acyclic impartial games on a finite number of positions, where perfect play is possible. Two players alternate in moving; they follow the same game rules; there is no chance device, no hidden information and there is a final position which determines the winner of the game. We play the normal version where a player who cannot move loses.

Move-size dynamic imitations

Definition

Suppose that the heap sizes in a two heap take-away game are a and b and that the previous player removed $0 < x$ tokens from the a-heap, and where $a + x \leq b$. Then the next player imitates the previous player if he removes x tokens from the b-heap.

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Definition

In the impartial game of Imitation Nim the players move as in Nim, but imitations are not allowed. In k-Imitation Nim, at most $k - 1$ consecutive imitations by one and the same player are allowed.

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It follows: Each terminal position is P; the previous player will win if and only if the position is P.

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The game is maybe more known as the impartial game "Corner the Queen" (Rufus P. Isaacs, 1960), where the two players alternate in moving one single Queen-of-Chess, aiming to get her to the lower left corner of a (large) chessboard; taking into consideration that, by moving, the distance to this corner must decrease.

Adjoining P-positions as moves

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The Wythoff-pairs

The P-positions of the game of Wythoff Nim, denoted by $\mathcal{P}_W = \{(a_n, b_n), (b_n, a_n)\}$, can be computed recursively by a certain "minimal exclusive" rule:

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Theorem

For $n \geq 0$, $a_n = mex\{a_i, b_i \mid i < n\}, b_n = a_n + n$.

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Suppose that $0 < \alpha < \beta$ *are positive real numbers such that* $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. *Then* $(\lfloor i\alpha \rfloor)_{i=1}^{\infty}$ *and* $(\lfloor i\beta \rfloor)_{i=1}^{\infty}$ *are complementary if and only if* $1 < \alpha < 2 < \beta$ *are irrational.*

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Beatty sequences

We call $(\lfloor i\alpha \rfloor)_{i=1}^{\infty}$ a Beatty sequence (of modulus α) if $\alpha > 0$ is irrational.

Properties of Wythoff's sequences

Theorem (Wythoff, 1907)

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The proof uses a simple inductive argument. We let Alice and Bob illustrate the idea.

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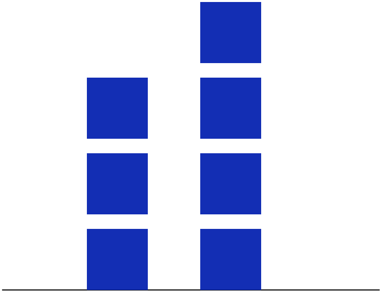

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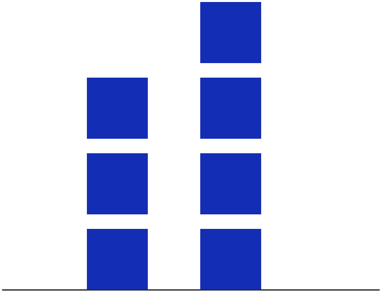

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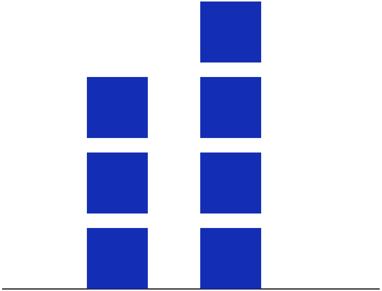

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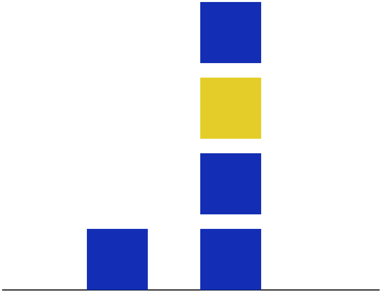

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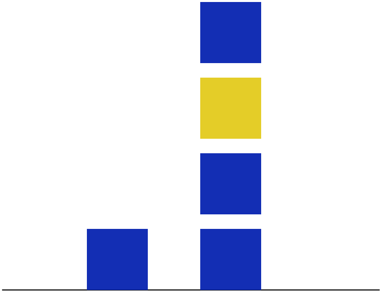

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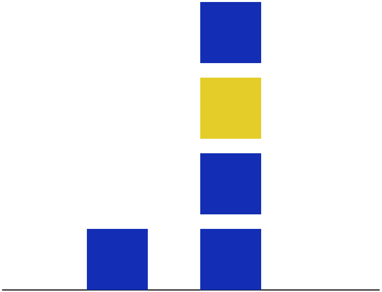

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- There are no other P-positions of Wythoff Nim as options. Induction gives the claim.

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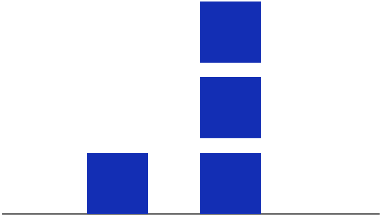

$\mathcal{P}_W =$
$(0,0),(1,2),(3,5),\ldots$

- Suppose our starting position is an *N*-position of Wythoff Nim, say $(3,4)$.
- We want to show that this position is an *N*-position of Imitation Nim.
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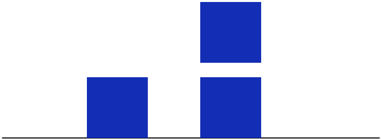
- ...Bob moves to $(1,3)$
- but Alice responds by moving to $(1,2)$ and wins.



2-Imitation Nim

Example:

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Notation:

The default color of a token is blue. A token is green if removal of it implies that an *imitation counter* is increased by one. A token is yellow if it may not be removed. The imitation counter is drawn as a black square.

Example: At most one imitation

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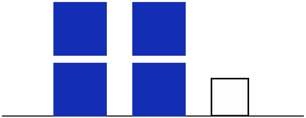

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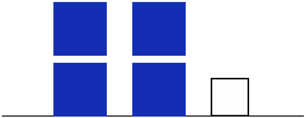

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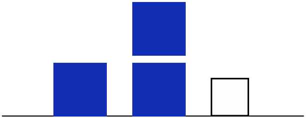

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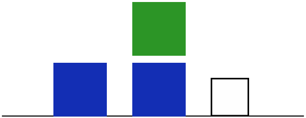

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- The starting position is $(2, 2)$.
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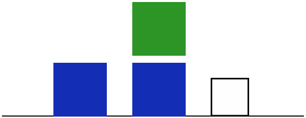

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- So $(2, 2)$ is an N-position for 2-Imitation Nim, unlike 2-pile Nim.

A Muller twist on Wythoff's game

Our next game is 2-Blocking Wythoff Nim. At each stage of game, the previous player may, before the next player moves, "block off" at most one diagonal option from the set of Wythoff Nim options. Any blocking maneuver is forgotten immediately after the move is carried out.

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A pair of tokens is painted red if it (together with the tokens on top), may not be removed. Notice that one of the red tokens may be removed, but not both at the same time.

Example: One diagonal option may be blocked

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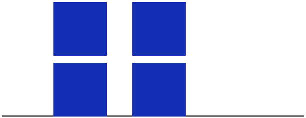

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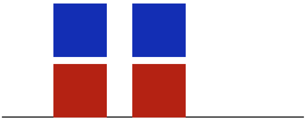

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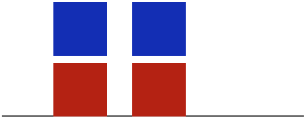

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- ...just as for 2-Imitation Nim.

The generalized algorithm

For fixed positive integers k and m and all $n \geq 0$, let

$$a_n = \operatorname{mex}\{a_i, b_i \mid i < n\}, b_n = a_n + \left\lfloor \frac{m}{k} n \right\rfloor .$$

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- If not, is the decision problem wether (x, y) is of the form (a_n, b_n) for some n tractable (polynomial in succinct input size)?

Fraenkel's game with a diagonal Muller twist

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III. but before the next player makes his move, the previous player may declare at most $k - 1$ of the diagonal options, i.e. with $i = j$, as blocked (Hegarty, Larsson 2006).

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Proposition (Hegarty, Larsson 2006, 2009)

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(vi) (a_i) *and* (b_i) *can be expressed by Beatty sequences only for special cases, namely if* $k \mid m$. *(Note: if* $k = 1$ *we have Fraenkel's m-Wythoff Nim.)*

A polynomial time approach

In the Appendix of my paper '2-Pile Nim with a restricted number of move-size imitations', P. Hegarty shows that if $k > 1$ and $m = 1$, the sequences are “close to” Beatty sequences with

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Via the general bound, for all n,

$$\lfloor (n - k - 1)\alpha \rfloor \leq a_n \leq \lfloor n\alpha \rfloor,$$

a polynomial time algorithm to determine $\mathcal{P}_{W_{k,m}}$ is developed by Udi Peled in his master thesis "Polynomializing seemingly hard sequences using surrogate sequences" (advisor A. S. Fraenkel) + paper. This has been generalized further by V. Gurvich (2012), to non-complementary sequences generated by a generalized mex-function.

A dynamic counting of P-positions of $W_{k,m}$

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Notice that at each stage of game, for a previous player winning strategy, at least $\xi((a, b))$ positions must be blocked off.

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Definition

Let m be a positive integer. Suppose that the previous player removed x tokens from a smaller (or equal) pile. Then if the next player removes $x + i$ tokens from the other pile, where $0 \leq i < m$, he *m-imitates* (or imitates) the previous player's move.

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In particular, $L(X) = k - 1$ if X is a starting position.

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Note: At the Integers 2007 conference, A. Fraenkel suggested the name "Limitation Nim" for the case $k = 1$.

Comparing the number of options for modified games

Let $k, m \in \mathbb{Z}_{>0}$. How does the number of options of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim vary as we alter k and m?

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Main Theorem

Theorem (Larsson 2009)

Let $k, m \in \mathbb{Z}_{>0}$.

- *Then* (x, y) *is a P-position of* (k, m)*-Imitation Nim if it is a P-position of* (k, m)*-Blocking Wythoff Nim and* $k - 1 \geq L(x, y) \geq \xi(x, y) \geq 0$. *In particular this holds for* (x, y) *a starting position of Imitation Nim.*

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- *Then* (x, y) *is a P-position of* (k, m)*-Imitation Nim if it is a P-position of* (k, m)*-Blocking Wythoff Nim and* $k - 1 \geq L(x, y) \geq \xi(x, y) \geq 0$. *In particular this holds for* (x, y) *a starting position of Imitation Nim.*
- *The "dynamic" P-positions of* (k, m)*-Imitation Nim are of the form* $(x, y + z)$ *where* (x, y), $x \leq y$, *is an upper P-position of* (k, m)*-Blocking Wythoff Nim and where* $-1 \leq L(x, y) < \xi(x, y) \leq k - 1$. *(Similar for "lower" positions.)*

Main Theorem

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- *Any other position of Imitation Nim is an N-position.*

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- Wythoff Nim can be viewed as the game where we to 2-pile Nim adjoin the P-positions as options.
- Limitation Nim $=$ $(1,1)$-Imitation Nim is the game where the next player may not imitate the previous player's most recent move. This game has the same P-positions as Wythoff Nim, if one only regards the starting positions.

Summary

- If we put a Muller twist to a game of Wythoff Nim, where we allow the previous player to block off at most $k - 1 \geq 0$ of the next player's diagonal options, then, regarded as starting positions, we get identical P-positions as for k-Imitation Nim. For the latter game, at most $k - 1$ consecutive imitations from one and the same player is permitted.

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- A "wider m-diagonal" of Wythoff Nim corresponds to an "m-relaxed" notion of an imitation. What is more, there is a precise dynamic correspondence between the winning positions of the games (k, m)-Blocking Wythoff Nim and (k, m)-Imitation Nim. This relationship constitutes our main theorem.

Question

- Do limited imitations and Muller twists also have some interesting interpretations for classical (bimatrix) games?