A golden lower bound for Property W sets

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- These type of estimates often concern bounds on the upper asymptotic density of sets, given certain avoidance criteria

The A and B sets

Let A denote any infinite set of positive integers. Let B denote its complement intersected with the positive integers. Then A and B are complementary sets on the positive integers. That is $A \cup B = \mathbb{N}$ and $A \cap B = \emptyset$.

The A and B sequences

We identify the set A with the unique sequence $A = (a_n)_{n=1}^{\infty}$ of strictly increasing positive integers. We are looking for an ordering of the elements in B that, together with the given A-sequence, satisfies a certain Property W.

Property W for a pair of sequences

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(2) The pair of sequences (A, B) satisfies Property W, if (1) holds and in addition, for all $i, j \in \mathbb{N}$, $\delta_i = \delta_j$ implies $i = j$

Property W for a set

The set A satisfies Property W if (2) holds; that is, if A's complementary set B is sufficiently distanced from A in this precise sense.

A W-impossible case

δ_n	0	1	2	2	3	5	6	$\cdots$
b_n	0	2	5	6	10	13	15	$\cdots$
a_n	0	1	3	4	7	8	9	$\cdots$
n	0	1	2	3	4	5	6	$\cdots$

Given the first few elements of the set $A = \{a_i\}_{i>0}$, there is no ordering of the elements in B, satisfying Property W. (For later use, let $b_0 = a_0 = 0$.)

How dense must a W-possible A-set be?

δ_n	0	1	2	3	4	5	6	$\cdots$
b_n	0	2	5	7	10	13	15	$\cdots$
a_n	0	1	3	4	6	8	9	$\cdots$
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Let ϕ denote the golden ratio. Wythoff Nim's upper P-positions are $(0,0),(1,2),\ldots,(a_n,b_n),\ldots$, where for all $n \in \mathbb{N}$, $a_n = \lfloor \phi n \rfloor$ and $b_n = \lfloor \phi^2 n \rfloor$. The consecutive differences $\delta_n = b_n - a_n$ are the natural numbers in strictly increasing order, that is $\delta_n = n$ for all n. Hence $\{a_i\}$ satisfies Property W.

$(1,2)$-GDWN produces interesting sequences

- 2-player impartial games: Nim is a famous normal play heap game, alternating play. Take any number of tokens from precisely one heap, at most the whole heap, finitely many heaps. A player who cannot move loses.

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- Wythoff Nim's moves are as in 2 heap Nim, or instead remove the same number from each heap.
- (1,2)-GDWN's rules are: move as in Wythoff Nim, or instead remove $t > 0$ tokens from one heap and $2t$ tokens from the other, only limited by the number of tokens in each heap.

The initial P-positions of $(1,2)$-GDWN

δ_n	0	2	4	1	3	6	8	$\cdots$
b_n	0	3	6	5	10	14	17	$\cdots$
a_n	0	1	2	4	7	8	9	$\cdots$
n	0	1	2	3	4	5	6	$\cdots$

Note that neither (b_n) nor (δ_n) is increasing. By the rules of game it follows that $\{a_i > 0\} \cup \{b_i > 0\} = \mathbb{N}$, $\{a_i > 0\} \cap \{b_i > 0\} = \emptyset$, and property W holds.

Comparing the entries of lower sequences

x_n	0	1	3	4	7	8	9	$\cdots$
a_n	0	1	2	4	7	8	9	$\cdots$
A_n	0	1	3	4	6	8	9	$\cdots$
n	0	1	2	3	4	5	6	$\cdots$

The x_n entries represent our W-impossible lower sequence, a_n GDWN and A_n Wythoff Nim. Ah, they look so similar! How can we distinguish some interesting behavior?

Detour: moves and P-positions in the first quadrant

y

pile 1

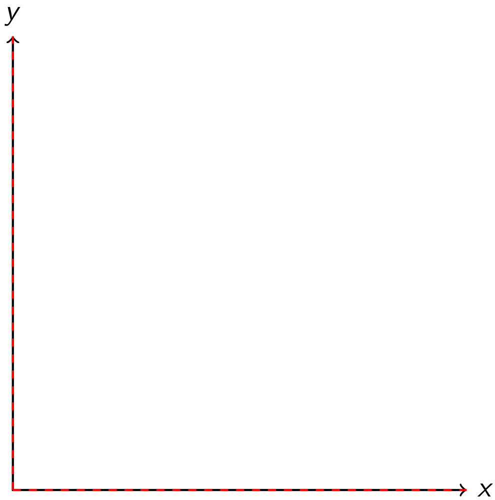
Moves will be translation invariant integer vectors
to subtract from given positions

P-positions (P-beams) are recursively defined
given the moves

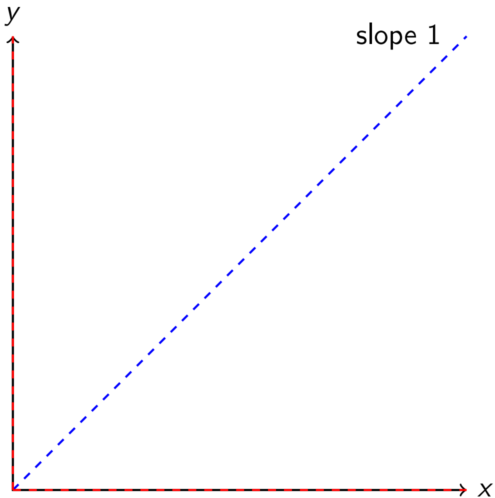
pile 2

x

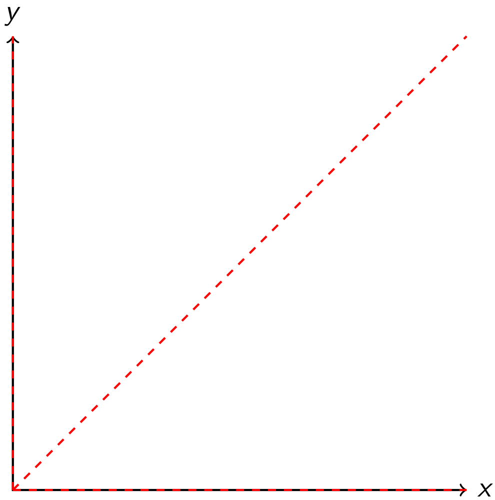
Nim's moves, $(0, t), (t, 0)$, $t \in \mathbb{N}$



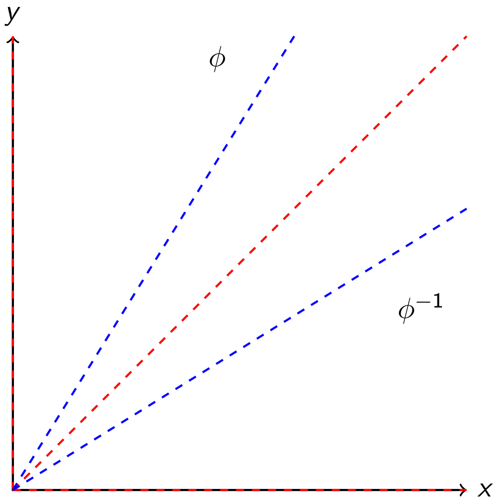
Nim's moves and its single P-beam



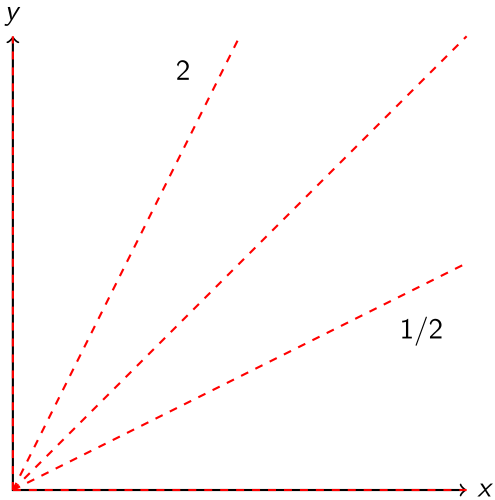
Wythoff Nim's moves



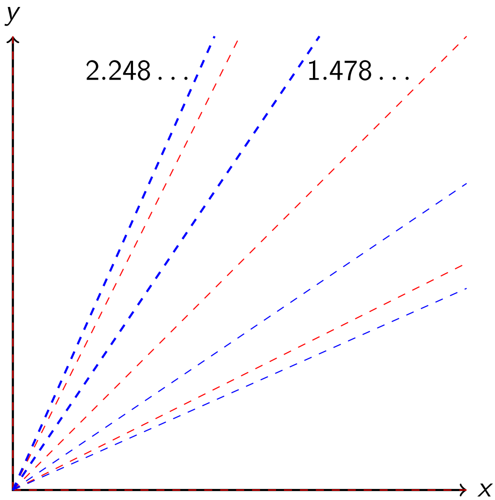
Wythoff Nim's moves and its splitted P-beams



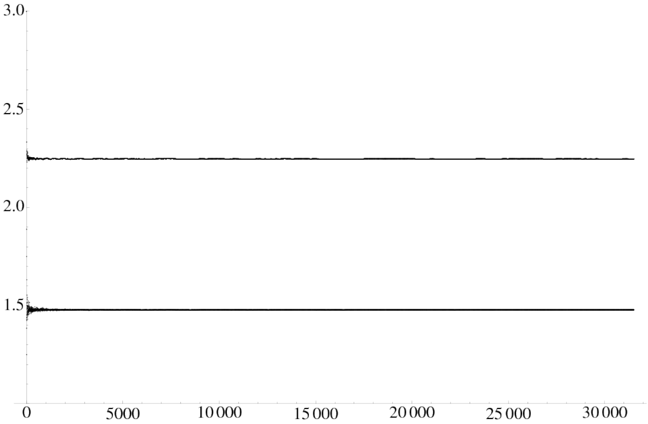
$(1, 2)$-GDWN's moves



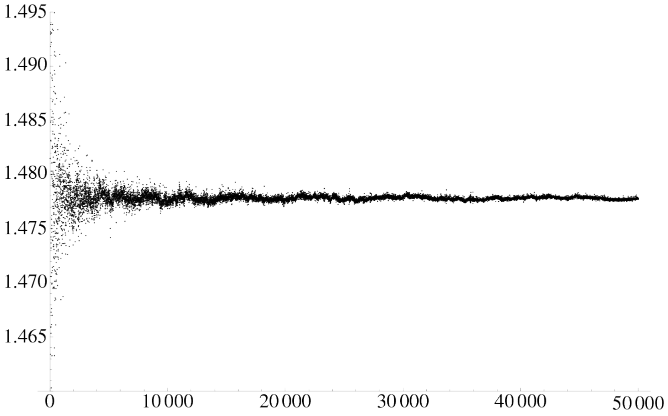
(1, 2)-GDWN's moves and P-beams experimentally



(1, 2)-GDWN's sequence of b_i/a_i



(1, 2)-GDWN's lower subsequence of b_i/a_i



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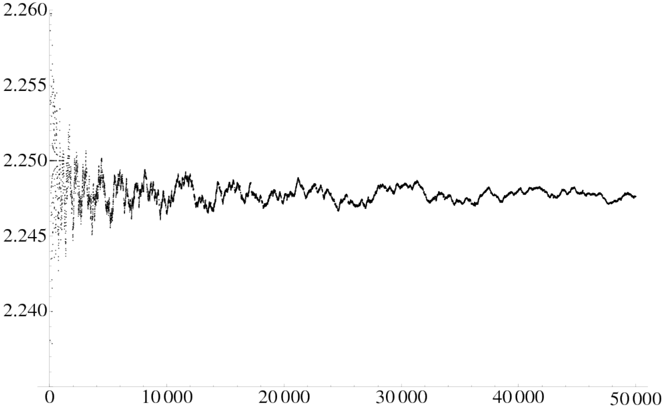
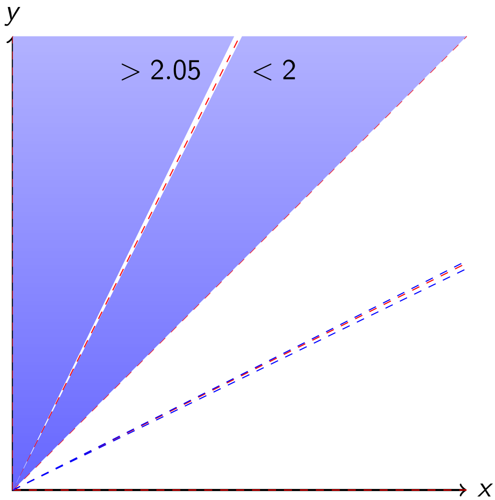


Θm: $(1, 2)$-GDWN's upper P-beams $(2, 0.05)$-split



Wythoff Nim extensions and Property W

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- A game is a Wythoff Nim extension, if we can define its set of P-positions as $\{(a_i, b_i), (b_i, a_i)\}$, with (a_i) increasing, $\{a_i\}$ and $\{b_i\}$ complementary, and such that $\{a_i\}$ satisfies Property W.

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- The result on the previous slide is made possible by bounding the lower asymptotic density of any a-sequence satisfying Property W.
- A game is a Wythoff Nim extension, if we can define its set of P-positions as $\{(a_i, b_i), (b_i, a_i)\}$, with (a_i) increasing, $\{a_i\}$ and $\{b_i\}$ complementary, and such that $\{a_i\}$ satisfies Property W.
- Observation: The game $(1, 2)$-GDWN is a Wythoff Nim extension.

Why a split? Explanation of Detour

Lemma

Consider $(1,2)$*-GDWN. Suppose, for* $n \in \mathbb{N}$,

$$\frac{\#\{i > 0 \mid a_i < n\}}{n} \geq \phi^{-1} - o(1).$$

Then the upper P-positions split.

Take a larger view

Theorem (Property W)

Suppose that $\{a_i\}$ satisfies Property W. Then, for $n \in \mathbb{N}$,

$$\frac{|\{i > 0 \mid a_i < n\}|}{n} \geq \phi^{-1} - o(1) \tag{1}$$

and

$$\frac{|\{i > 0 \mid b_i < n\}|}{n} \leq \phi^{-2} + o(1). \tag{2}$$

In particular the result holds for $\{(a_i, b_i)\}$ representing the upper P-positions of any Wythoff Nim extension.

Proof

- Define the y-sequence as the unique permutation of a given b-sequence, with entries in increasing order. That is $y_n < y_{n+1}$ for all n and $\{y_n\} = \{b_n\}$.

Proof

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- Define the unique surjective index-function $j : \mathbb{N} \to \mathbb{N}$, $j = j(n)$ such that, for all n, $a_j \leq n < a_{j+1}$. (This is well defined by (a_i) strictly increasing and $a_1 = 1$.)

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- We get $\frac{1}{\phi^{-1}-\epsilon'} < \frac{a_{j(n)}}{j(n)}$, which implies that there is an $\epsilon > 0$ such that, for all sufficiently large n, $\phi n + \frac{\epsilon n}{2} < a_n$.

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- We get $\frac{1}{\phi^{-1} - \epsilon'} < \frac{a_{j(n)}}{j(n)}$, which implies that there is an $\epsilon > 0$ such that, for all sufficiently large n, $\phi n + \frac{\epsilon n}{2} < a_n$.
- By complementarity this implies, for all sufficiently large n, $\phi^2 n - \gamma(\epsilon) \geq y_n$, where $\gamma(\epsilon) > \frac{\epsilon n}{2}$ is a function of ϵ only.

Thus

$$\begin{aligned}\delta'_n &:= y_n - a_n \\ &< (\phi^2 - \phi)n - \epsilon n \\ &= (1 - \epsilon)n,\end{aligned}$$

for all sufficiently large n. Hence, $(\delta'_n)_{n \le N}$ must contain at least ϵN (pairwise) repetitions, for all sufficiently large N.

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for all sufficiently large n. Hence, $(\delta'_n)_{n \le N}$ must contain at least ϵN (pairwise) repetitions, for all sufficiently large N. But this does not yet contradict Property W. We must show that for any b-sequence, some δ-repetition will be forced.

- Given $C \in \mathbb{N}$, define the finite set $S_b = S_b(C)$ of all indices of b-entries smaller than C.

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- Let (n_i) be the unique increasing sequence of the numbers in S_b.
- Then, $n_i \geq i$, for all i, and therefore also, by (a_i) increasing, $a_{n_i} \geq a_i$, for all i.

- Suppose now that N is sufficiently large, so that $(\delta'_n)_{n\leq N}$ contains ϵN repetitions, as defined in the previous paragraph, and study the unique set S_b of size N.

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- Thus, since $\sum_{S_b} a_i \geq \sum_{i=1}^N a_i$ and $\sum_{S_b} b_i = \sum_{i=1}^N y_i$, we get $\sum_{S_b} \delta_i \leq \sum_{i=1}^N \delta'_i$, and so the sequence $(\delta_i)_{i=1}^N$ must also contain at least ϵN repetitions for all sufficiently large N.

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- This contradicts property W, and so (1) must hold, and thus, by complementarity also (2). $\square$

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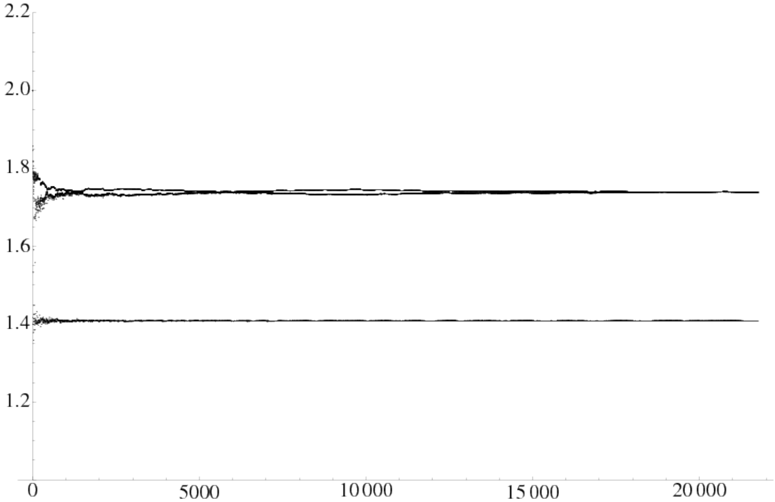
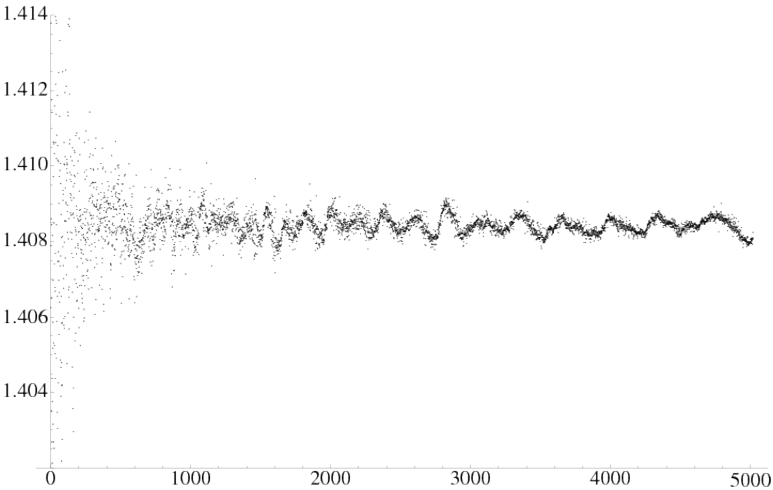
$(2,3)$-GDWN sequence b_i/a_i



$(2, 3)$-GDWN lower subsequence b_i/a_i



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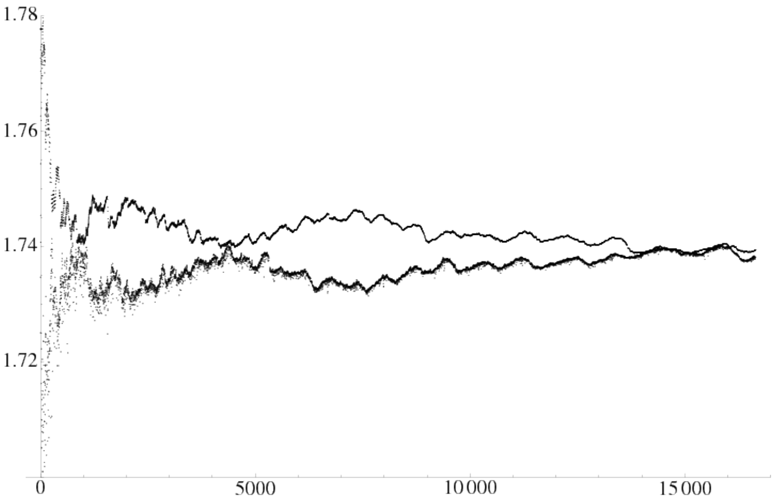
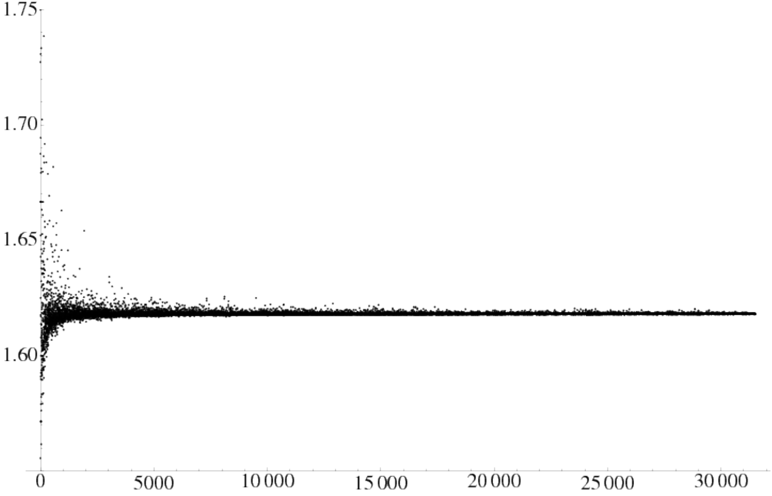
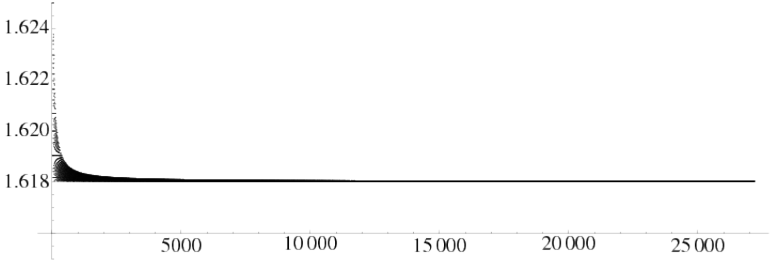


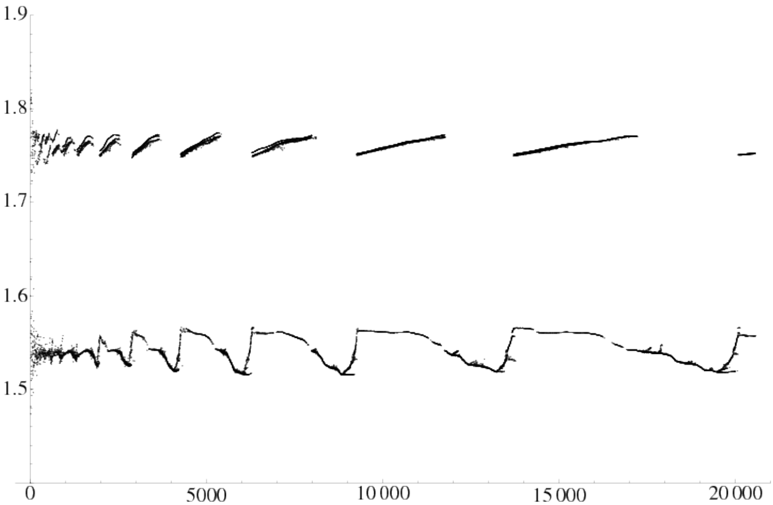
$(2,4)$-GDWN sequence $b_i/a_i \to \phi$?



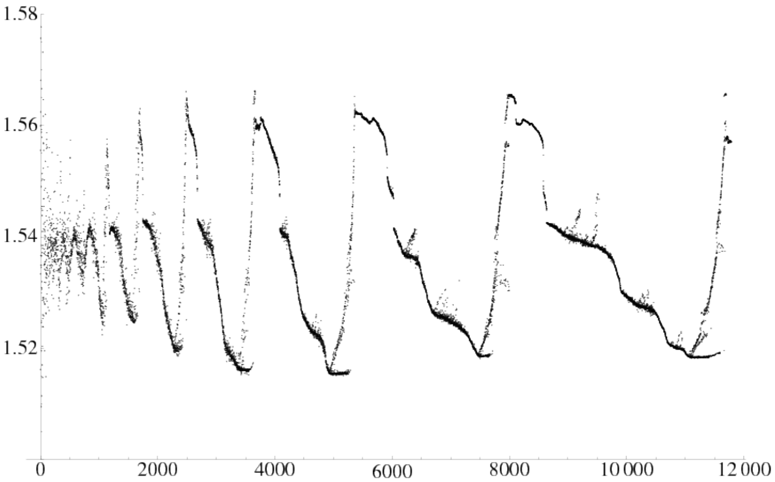
(p, q)-GDWN sequence for non-Wythoff pairs: $B_i/A_i \rightarrow \phi$



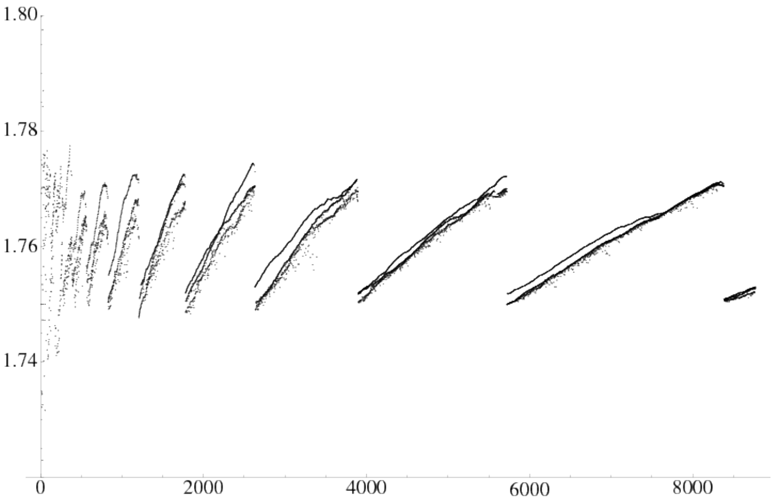
$(3,5)$-GDWN sequence b_i/a_i



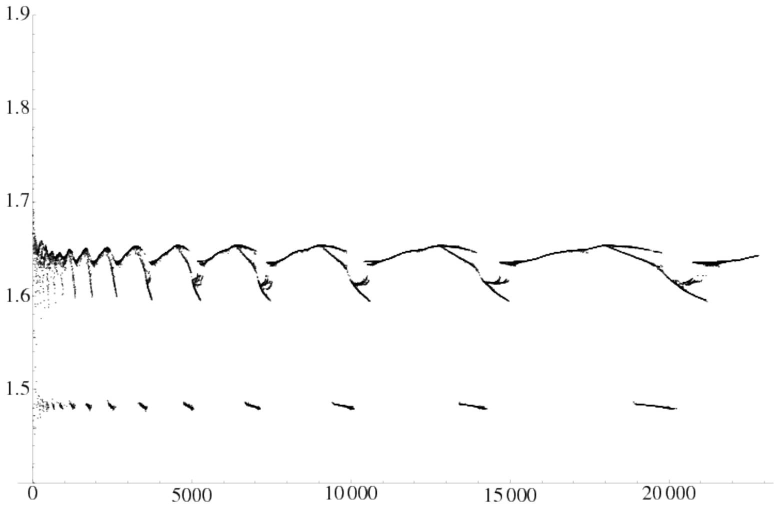
(3, 5)-GDWN lower subsequence b_i/a_i



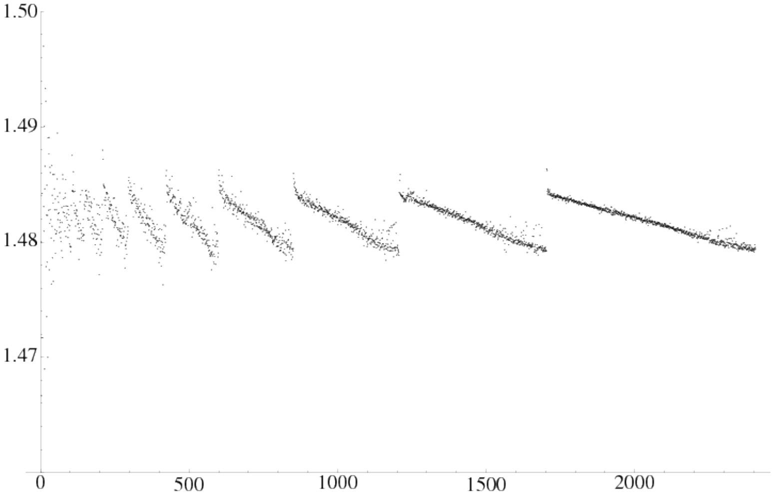
$(3, 5)$-GDWN upper subsequence b_i/a_i



(4, 6)-GDWN sequence b_i/a_i



(4, 6)-GDWN lower subsequence b_i/a_i



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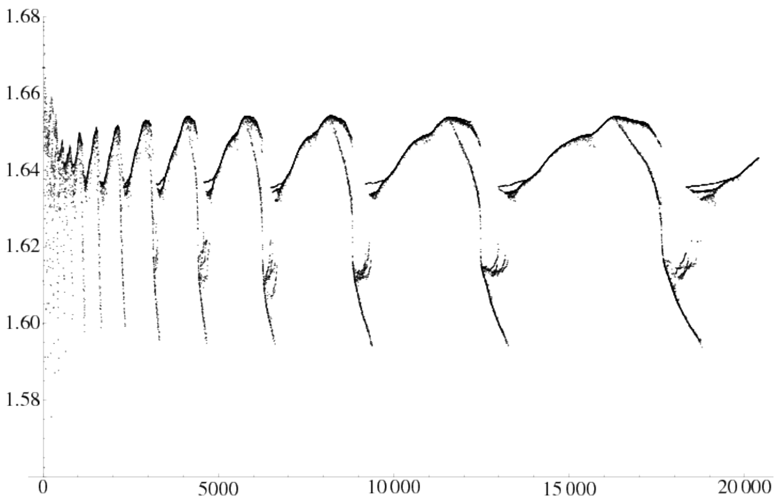
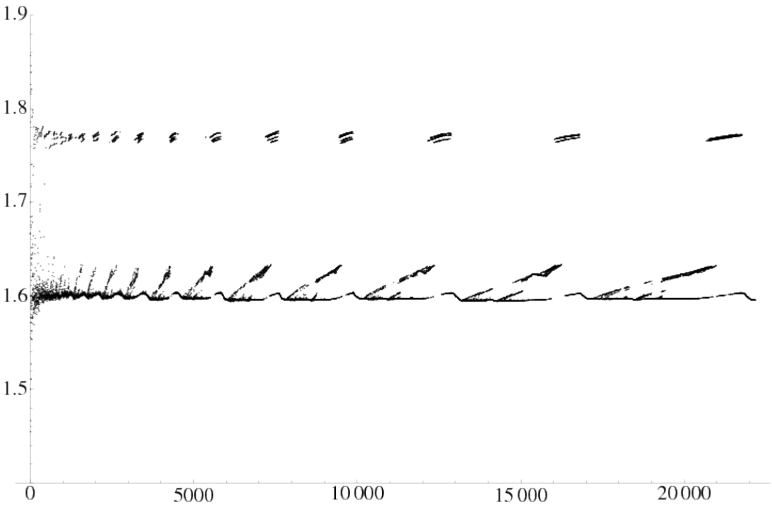
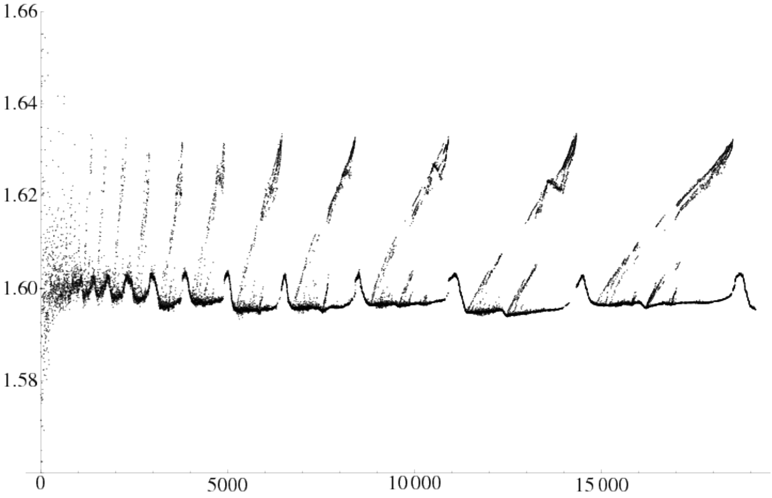


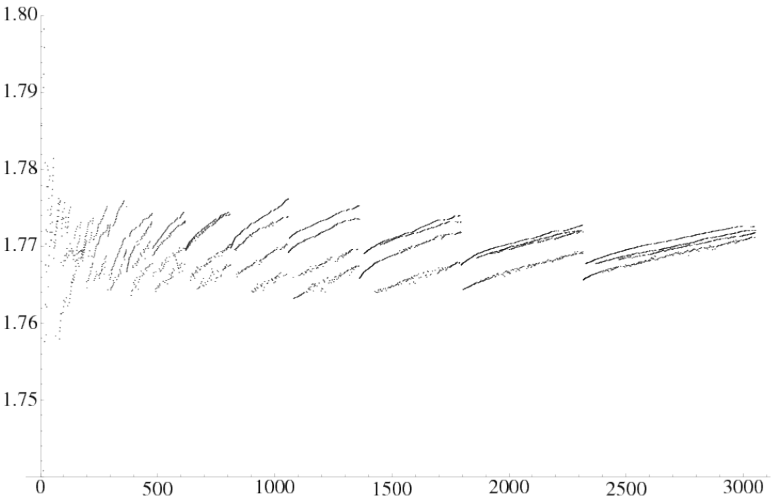
(4, 7)-GDWN sequence b_i/a_i



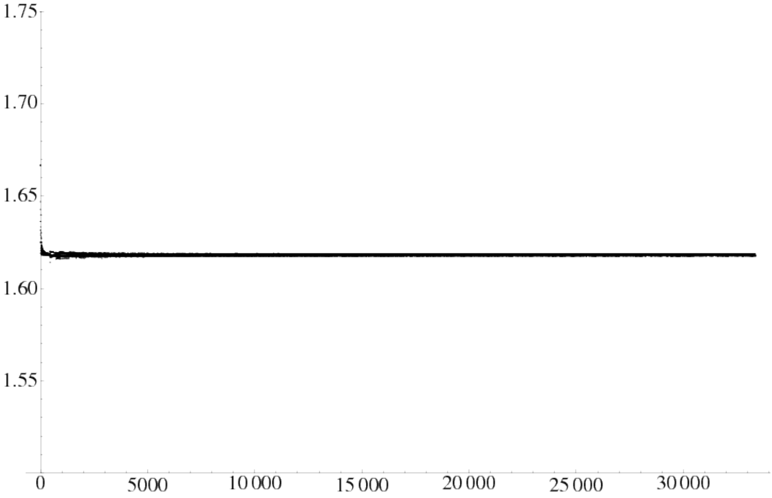
$(4, 7)$-GDWN lower subsequence b_i/a_i



$(4,7)$-GDWN upper subsequence b_i/a_i



$(731, 1183)$-GDWN sequence b_i/a_i (a Wythoff pair)



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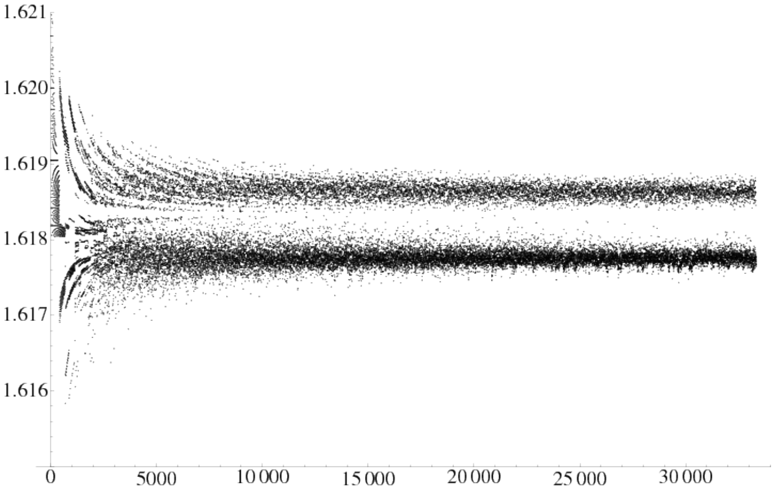

P-beams split for $(1,2)(2,3)(3,5)(5,8)$-GDWN?