Self organization in combinatorial games

MAM-seminar, UKK, MDH

Urban Larsson, Dalhousie University, Killam Trust, Halifax Canada,
with: Cook, Neary, Friedman, Landsberg, Garrabrant, Phipps-Morgan

2-player combinatorial games with perfect information

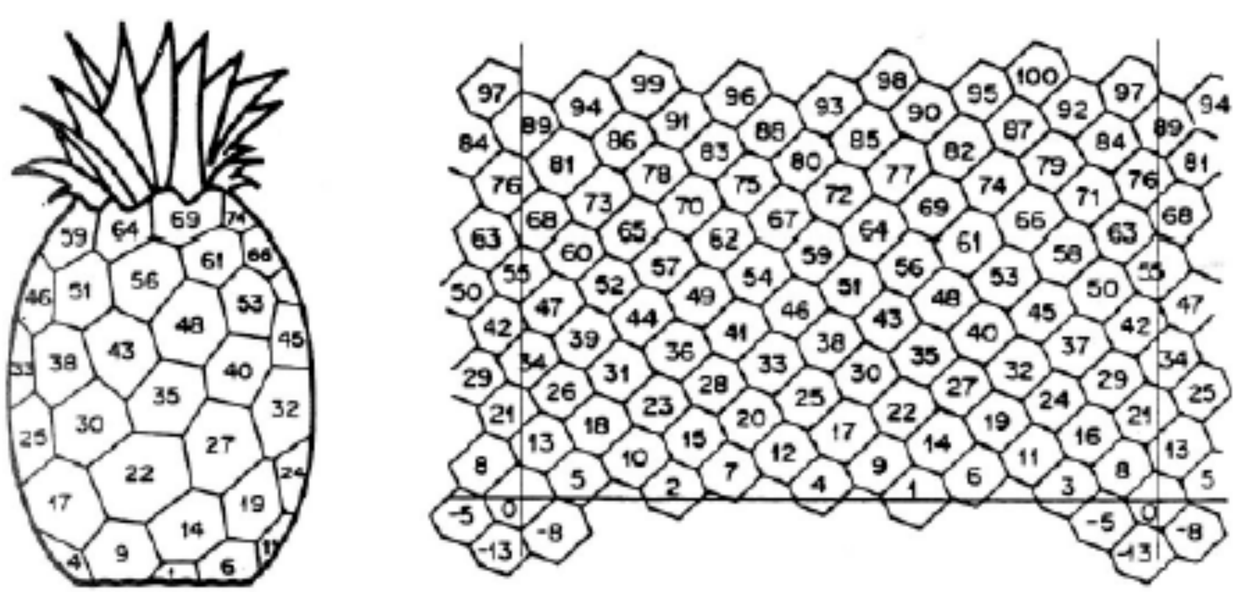
leads to theory of disjunctive sum of games

leads somewhere else

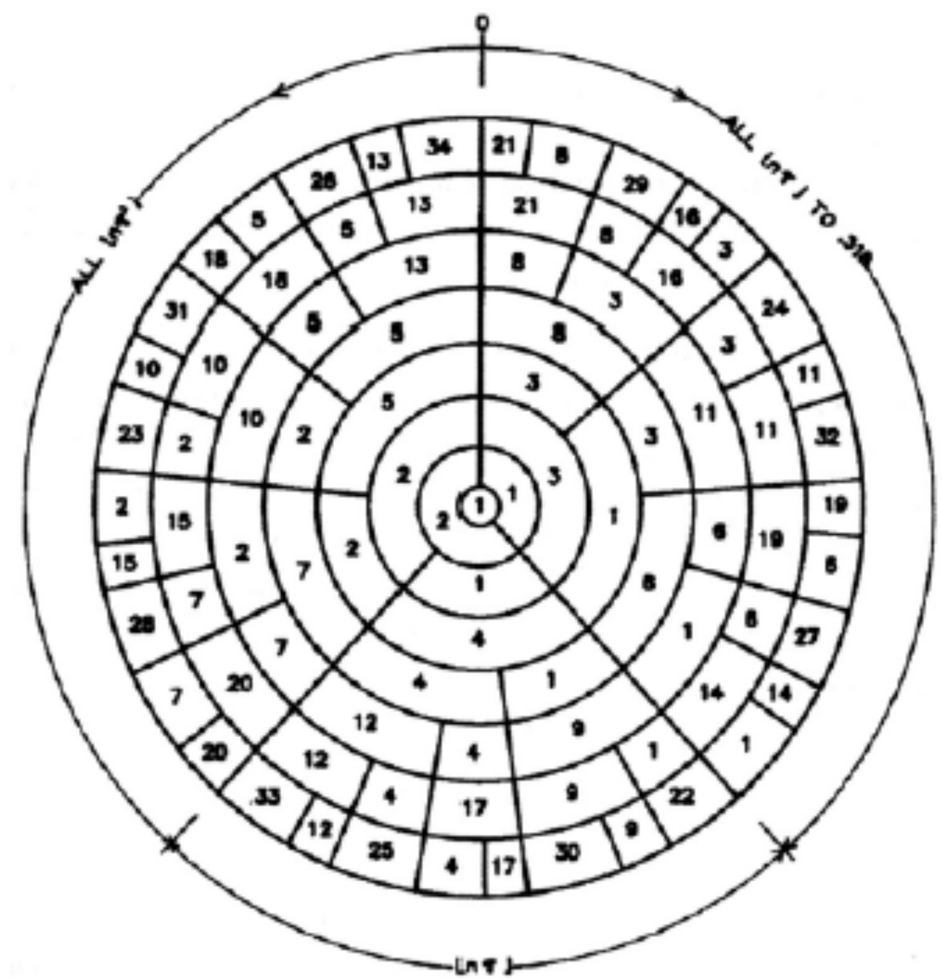
- Some games have mathematical appeal: Nim, Wythoff's Nim
- nice closed formulas (Xor-addition, Beatty sequences) for winning positions,
- or at least a log-polynomial algorithm.
- other games are thought to be harder: Chomp, Chess, Go.
- 3-row Chomp has geometry and can be analyzed via techniques adapted from physics (renormalization)
- Friedman and Landsberg (2007)

Some games hide in patterns of flowers and plants

COXETER, H. S. M. "The Golden Section, Phyllotaxis, and Wythoff's Game." Scripta Mathematica 19 (1953). KAPPRAFF, J., BLACKMORE, D. and ADAMSON, G. W. "Phyllotaxis as a dynamical system: a study in number", in Symmetry of Plants (eds. R. V. Jean and D. Barabe) (1997)



5-8 phyllotaxis in a pineapple



Adamson's Wythoff wheel

Why Wythoff's game and self-organization?

Adamson's wheel is related to phyllotaxis, the Zeckendorf maximal representation and to the P-positions of Wythoff Nim. Moreover, the author's write:

"… we describe Wythoff's game, which holds the key to describing phyllotaxis as a dynamical system."

Kappraff, Adamson, Blackmore

Impartial games on 2 heaps of tokens, last move wins

Ancient, also Bouton 1902

2-heap Nim: remove any number from exactly one of the heaps: who wins from (3,5)?

Ancient, also Wythoff 1907

Wythoff Nim: Nim or instead remove the same number from both heaps: who wins from (3,5)?

The classical Wythoff Nim's set of P-positions is $\{(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor), (\lfloor n\phi^2 \rfloor, \lfloor n\phi \rfloor)\}$ (W. A. Wythoff 1907), where n runs over the nonnegative integers and $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

Larsson 2012, 2014

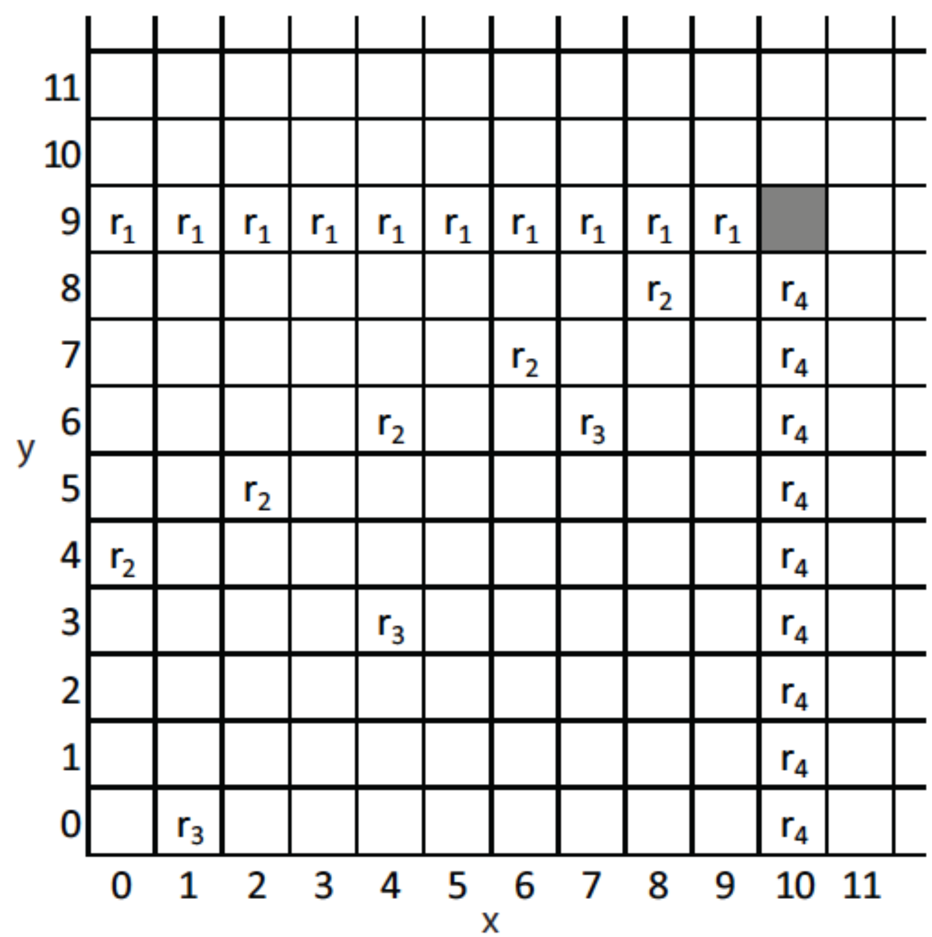
(p,q)-GDWN: Wythoff Nim, or instead remove a multiple of p from one heap and the same multiple of q from the other

Theorem: (1,2) and (2,3)-GDWN *split*

Linear Nimhoff

with Friedman, Landsberg, Garrabrant, Phipps-Morgan

R={(1,0),(0,1),(2,1),(3,3)}



Move from current position to an option on the line r_i

Study data for some

sets in Linear Nimhoff. (a) $R = \{(1,0),(0,1)\}$ (i.e., two-pile Nim); (b) $R = \{(1,0),(1,1),(0,1)\}$ (i.e., Wythoff), (c) $R = \{(1,0),(3,2),(1,1),(0,1)\}$; (d) $R = \{(1,0),(1,1),(2,3),(1,2),(0,1)\}$; (e) $R = \{(1,0),(1,1),(1,2),(0,1)\}$; (f) $R = \{(1,0),(1,1),(1,2),(1,8),(0,1)\}$

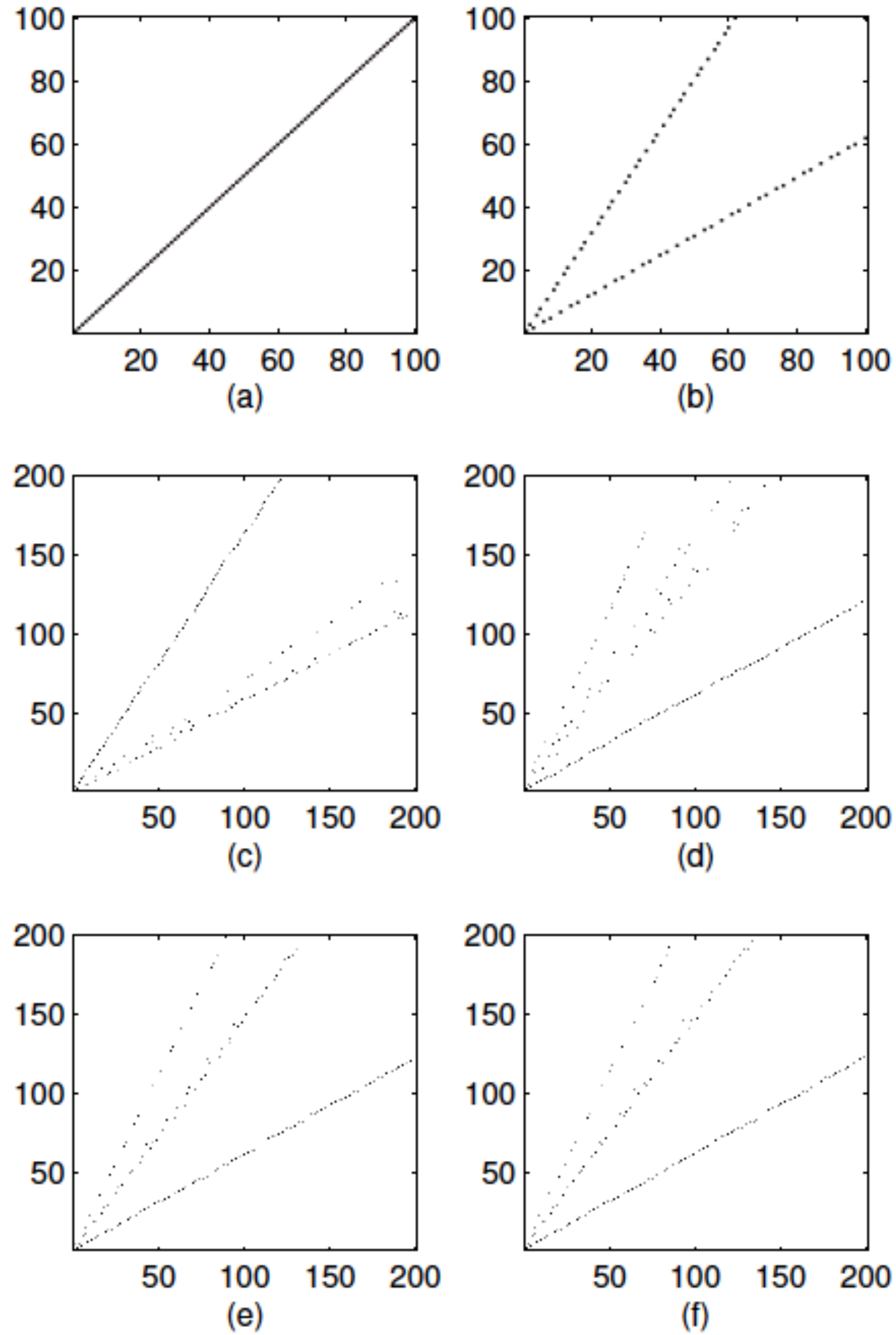
There is a lot of geometry in these games. What is a reasonable explanation of the observed behavior?

Losing positions (P-positions) for the current player

Nim

Wythoff Nim

Linear Nimhoff variations

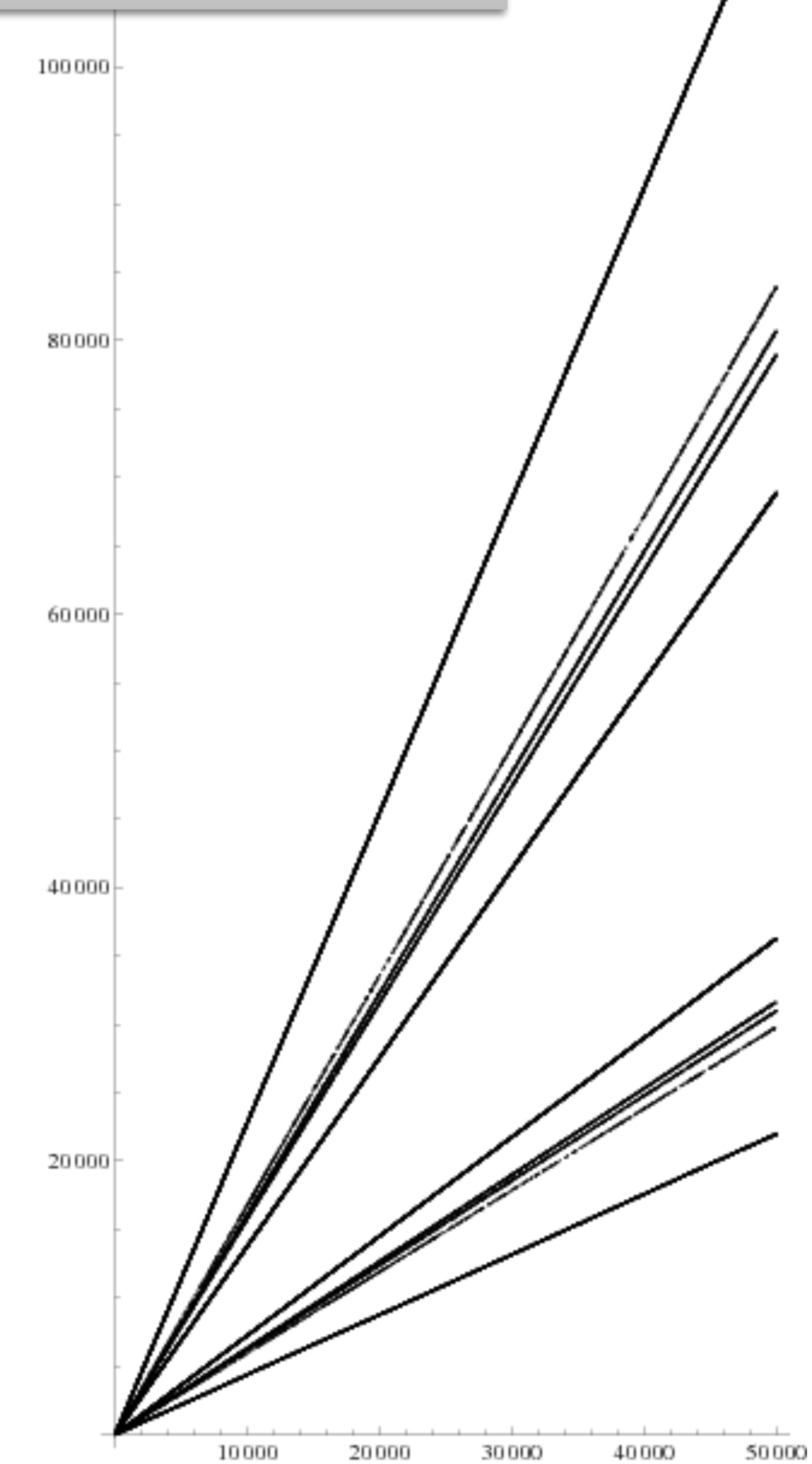


Assumption: "forbidden regions" no P-position between the *P-lines*, justified by all experimental data over the years

The game
{(1,2),(2,3),(3,5),(5,8)}-GDWN

GDWN has symmetric rules, and hence symmetric P-positions

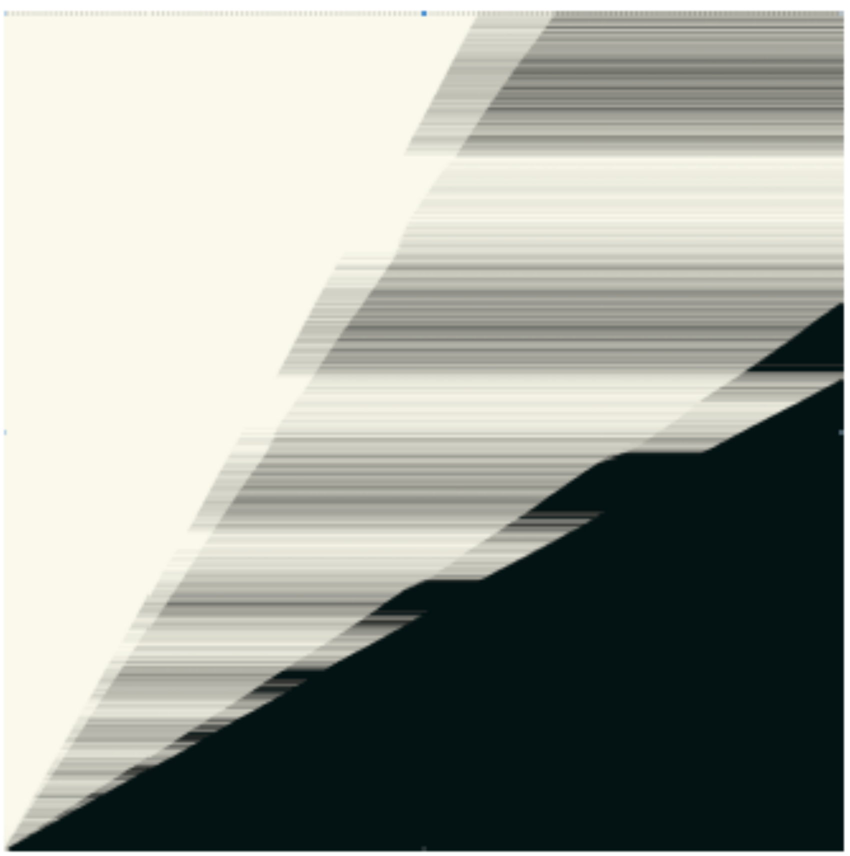
One rule fills each forbidden region with N-positions



Computations to 50000 for many games confirm hypothesis, P-lines are "splitting", and between them a *filling property*

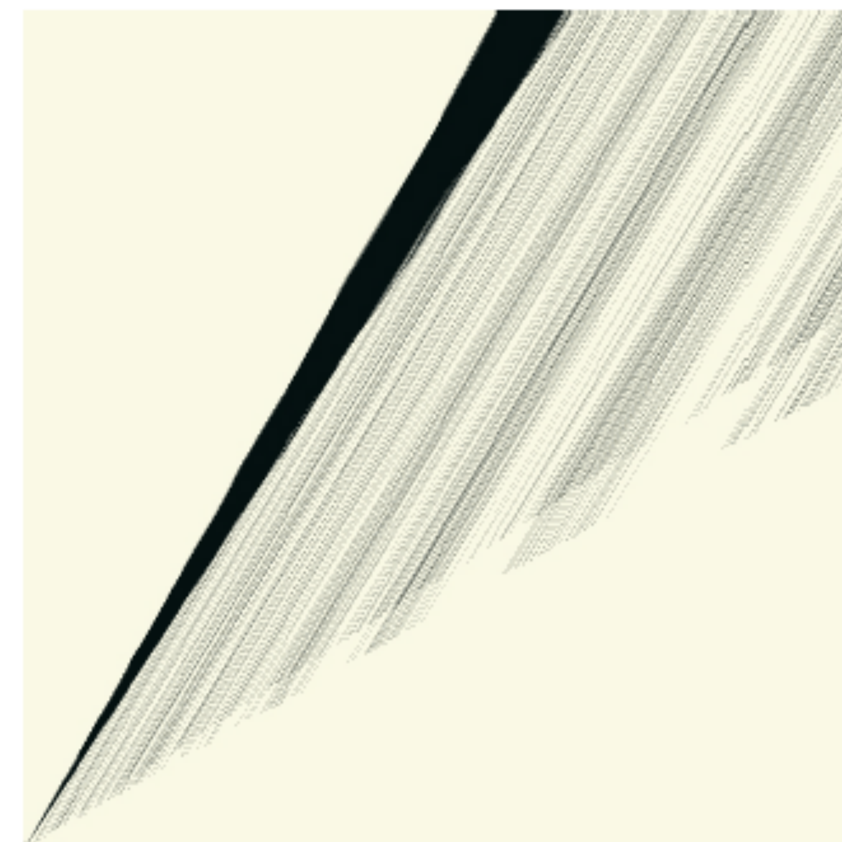
(3,5)-GDWN

One rule per picture, black lines/regions are parents to P-positions (N-positions) for this rule



horizontal Nim rule

diagonal Wythoff-type rule



(3,5)-rule

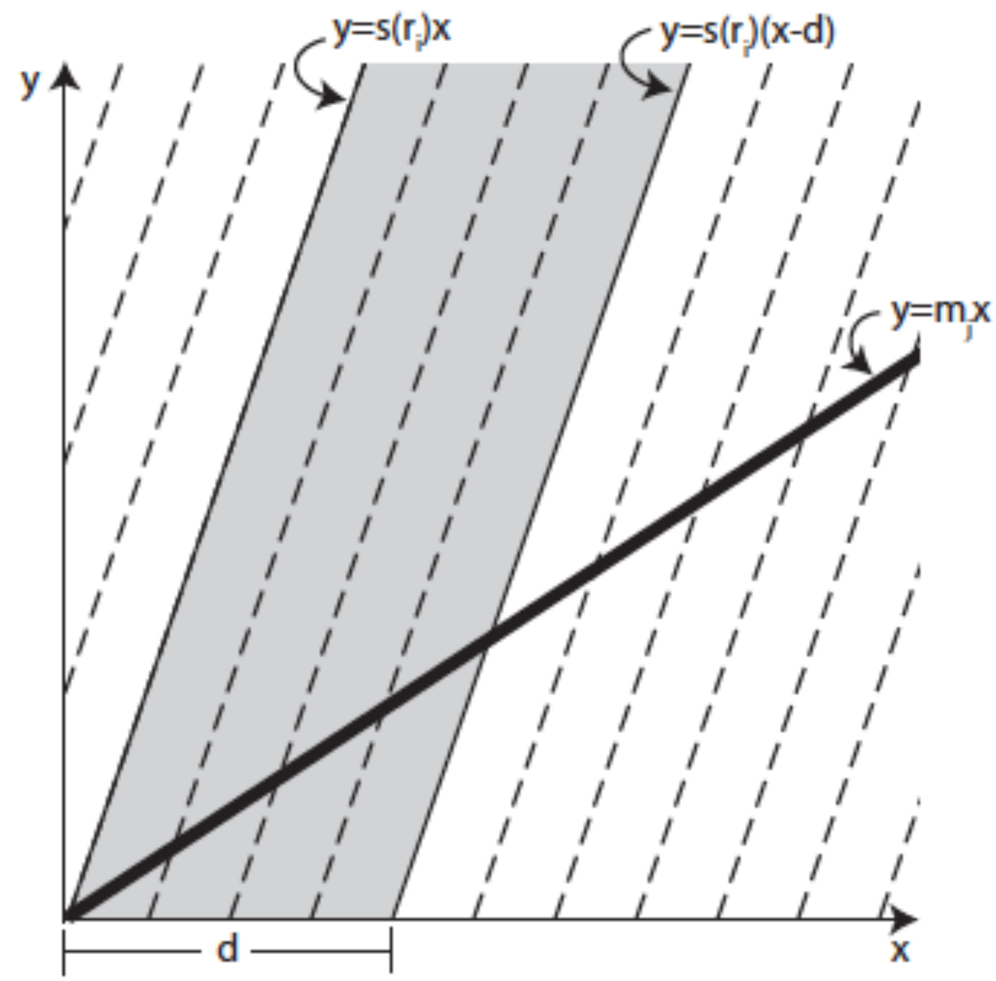
The starting point of our analysis is a sharpening of the empirical observation of forbidden regions, namely that *for each forbidden region, there exists a single rule $r \in R$ by which it is possible to move from any point in that forbidden region to a P-position*; in other words, all points within a given forbidden region are parents of P-positions under the *same* rule r (although they may also be parents under other rules as well).

C_r is the set of all equivalence classes (lines) given by the rule r.

Proposition: given a rule set R, consider a forbidden region F associated with a particular rule $r \in R$. We claim that parents under the rule $r \in R$ fill this forbidden region if and only if each set in C_r contains exactly one P-position.

This result implies an important density property.

Computing the fraction $f_{i,j}$ of P-positions contributed by a single P-line, for a given rule r_i



slope of $r_i = b_i/a_i$

density of P-line j is λ_j

slope of P-line j is m_j

$$f_{i,j} = \frac{\lambda_j}{b_i - m_j a_i}$$

The n+1 rules are ordered by increasing slope

The proposition implies a density property:

positive y-intercept yields, for fixed rule i,

$$\sum_{j=i}^{n} \frac{\lambda_j}{m_j a_i - b_i} = 1.$$

Positive x-intercept yields

$$\sum_{j=1}^{i-1} \frac{\lambda_j}{b_i - m_j a_i} = 1.$$

Altogether

$$\sum_{j=1}^{i-1} \frac{\lambda_j}{b_i - m_j a_i} = 1 \;\; \forall\, i \in \{2, 3, \ldots, n+1\},$$

$$\sum_{j=i}^{n} \frac{\lambda_j}{m_j a_i - b_i} = 1 \;\; \forall\, i \in \{1, 2, \ldots, n\}.$$

since each Nim rule is excluded from one of the sets of equations

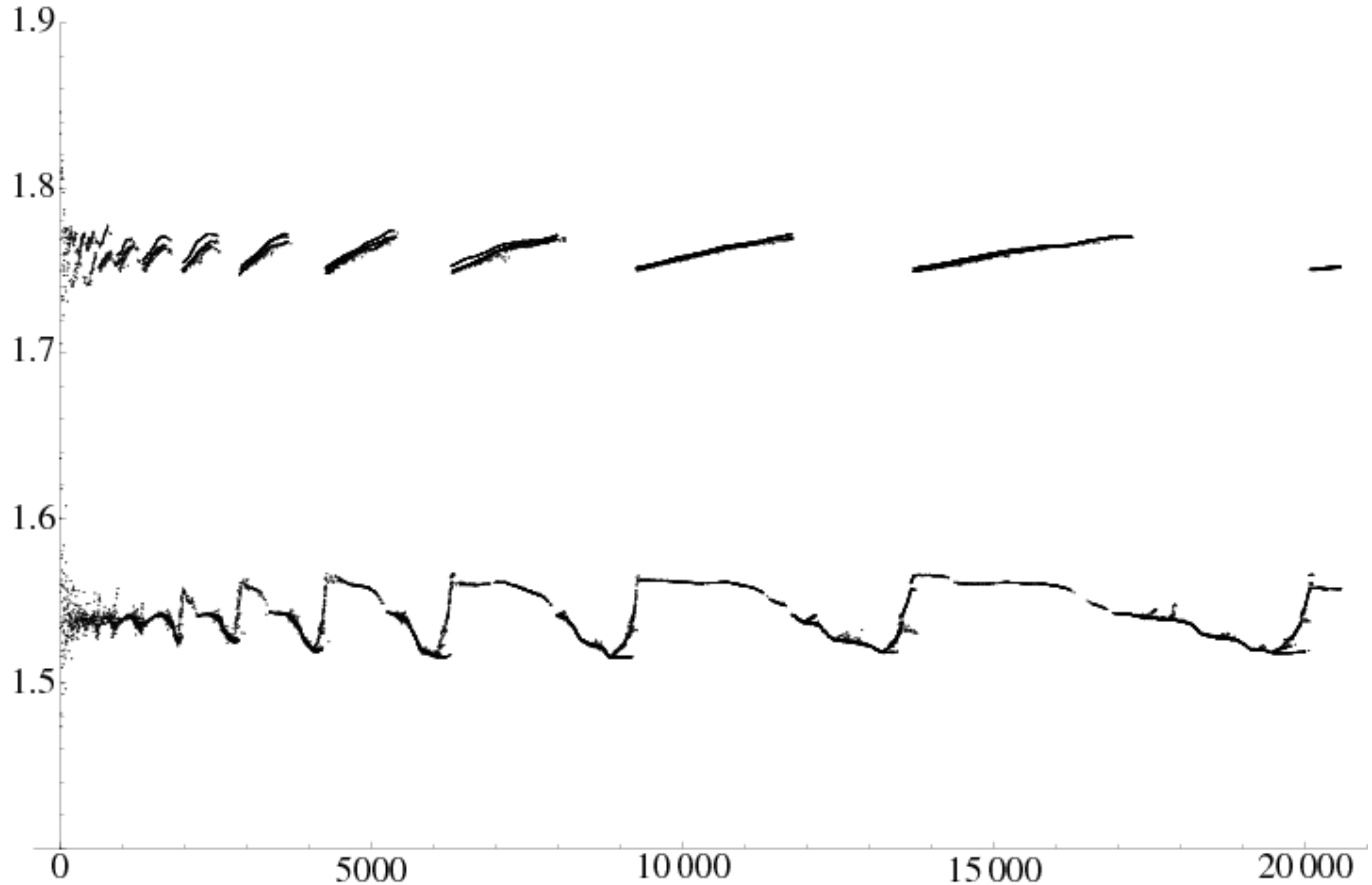
But this is not the whole truth, and after all, the reasoning is built on a non-rigorous method adapted from physics

The slopes of the upper P-beams of (p,q)-GDWN, many (p,q), show fluctuations far beyond the assumption of uniform distribution

Some P-lines are more interesting than a uniform distribution

The fill rule property appears to hold for (3,5)-GDWN, but there is also some quasi log-periodic pattern… the P-lines are not lines, rather beams, *P-beams*

Variations to the upper slopes of (3,5)-GDWN



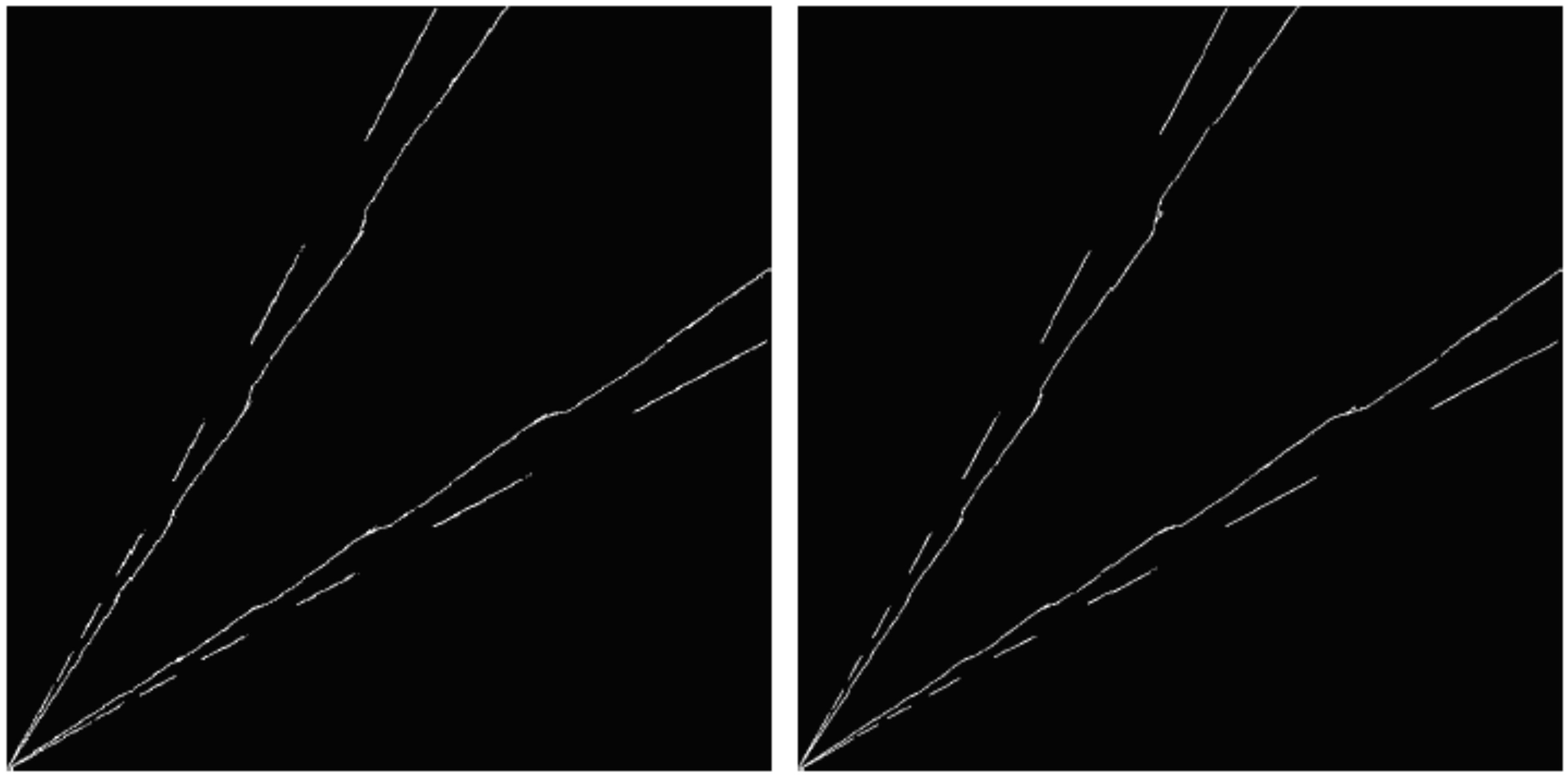
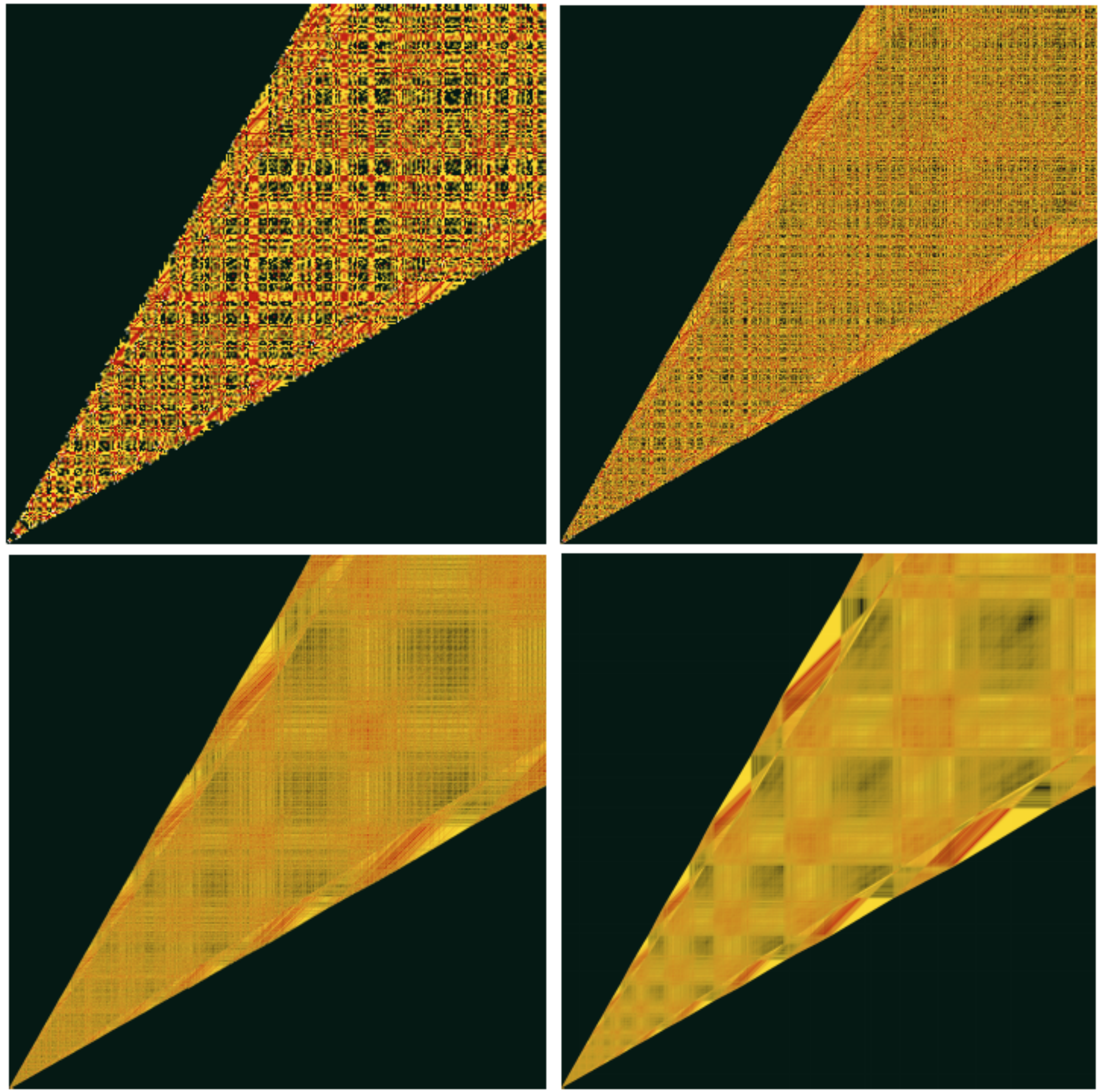


Figure 5: Left: P-positions of (3,5)GDWN for $x \leq 32600$, $y \leq 32600$. Right: P-positions of (3,5)GDWN for $x \leq 47800$, $y \leq 47800$; by a ≈ 1.478 scaling we obtain geometric invariance of P-positions.

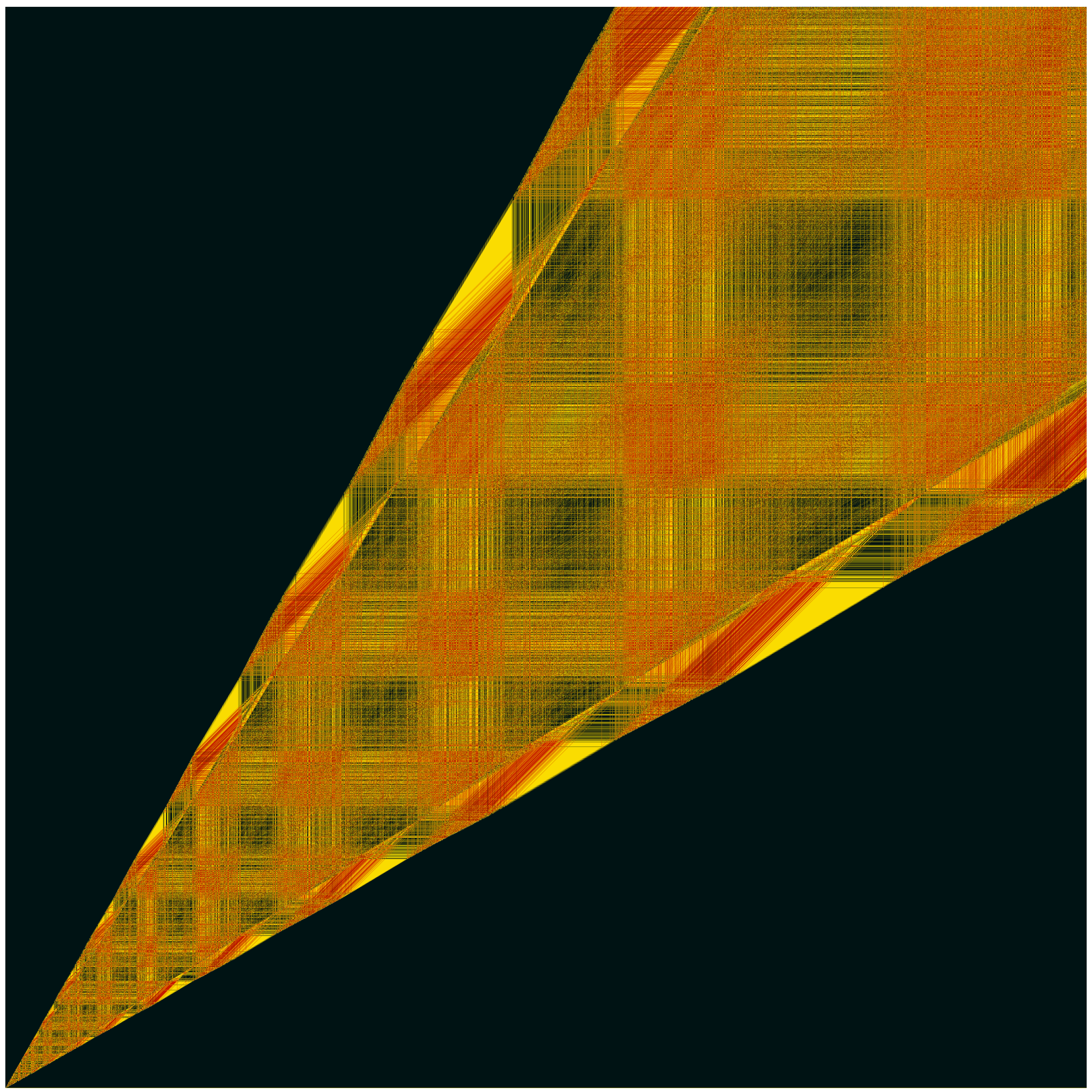
The colors represent different numbers of P-positions as options

The system self-organizes into visibly distinct regions, i.e. more geometry

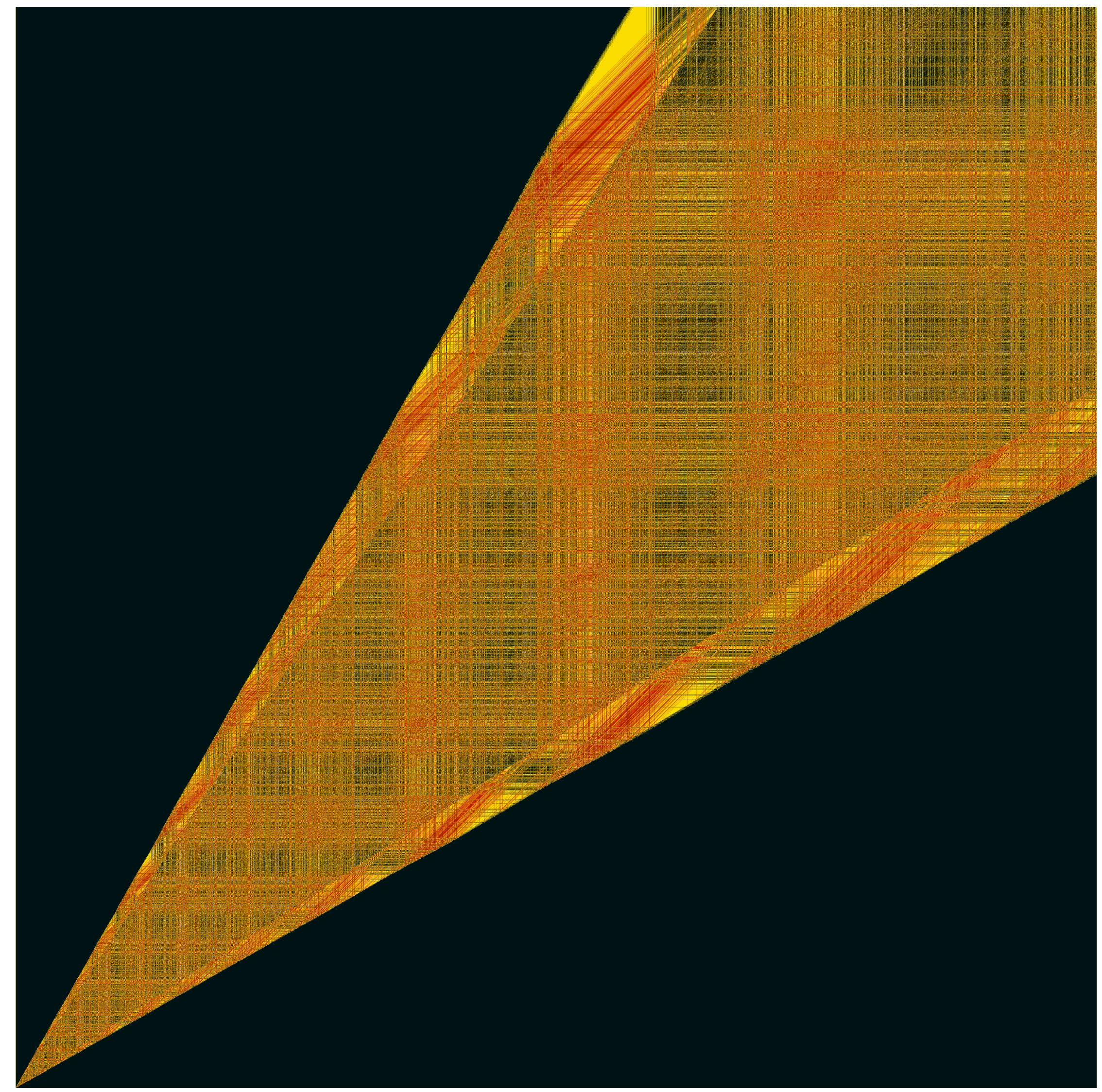


Distorted (3,5)-GDWN

(0,1) is forced P-position



(0,3) is forced P-position



Is (p,q)-GDWN stable to distortions?

Landsberg & Friedman: Chomp is, Nim is not (but scale invariance holds either way)

My computer does not open color pictures of 100000*100000 pixels, so instead I ran the code showing just the P-beams, and the familiar pattern begins to reappear

20000*20000

100000*100000

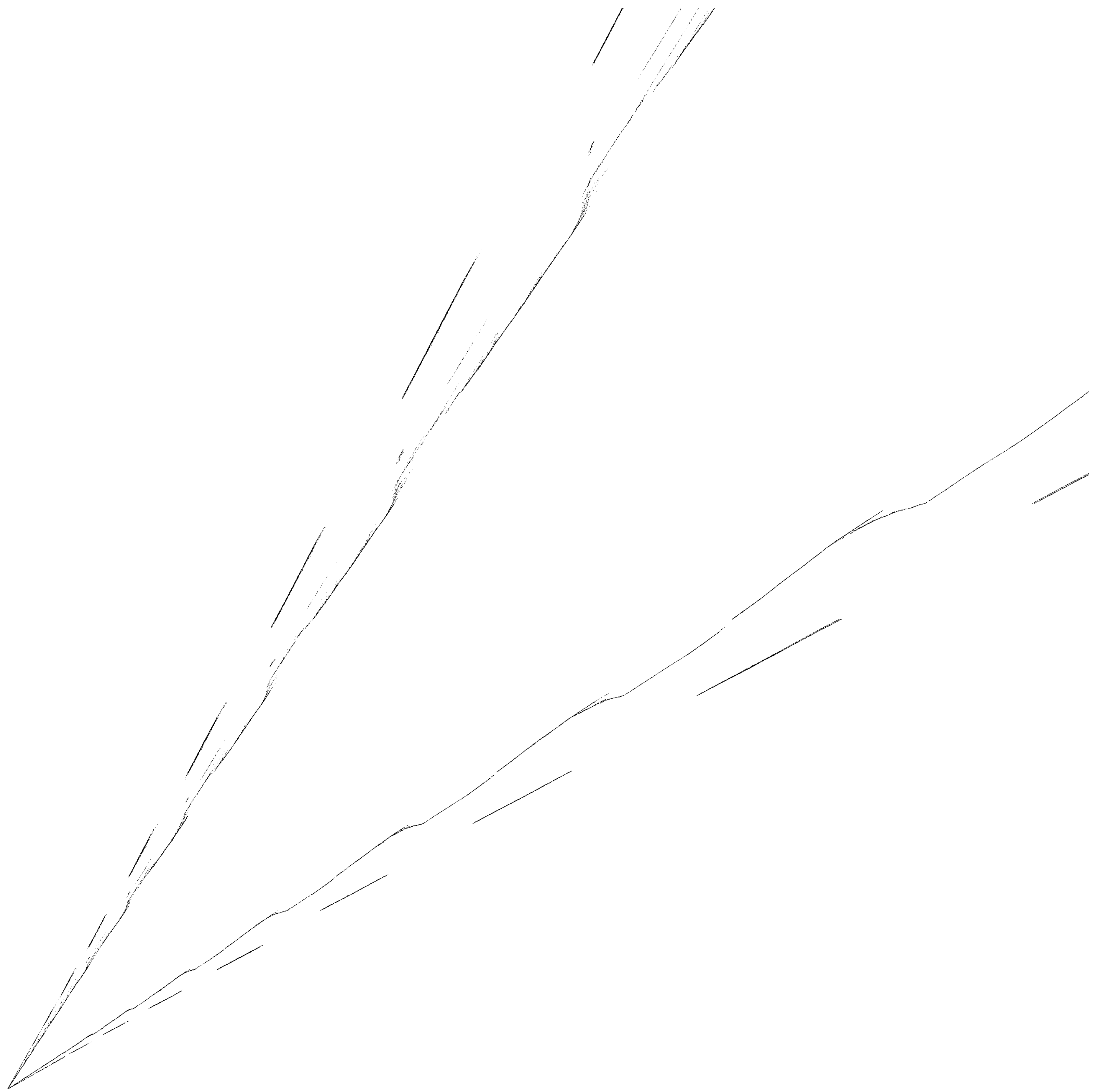
Each P-position is magnified to 10*10 pixels or more to be visible

R={(3,2)(2,1),(1,1),(0,1),(1,0)}: no fluctuation creating additional geometry for non symmetric games

Attempted conjecture: Scale invariant log-quasi-periodic fluctuations appear for Linear Nimhoff if and only if it is a (p,q)-GDWN game with (p,q) a Wythoff pair, or (p-1,q-1) a Wythoff pair, except for (p,q)=(1,2) or (2,3), and the fluctuations are not sensitive to small perturbations of P-positions.

(3,5)-GDWN with adjoined (4,7) move -> Linear Nimhoff: **no fluctuations**, but a new P-line and shifted mean slopes of the old ones. We have a density argument for when a new P-line appears, given an adjoined rule. But we are not sure if we can use it for games with fluctuations: here it seems OK.

Sometimes fluctuations remain, even though a new P-line appears: (3,5)-GDWN with adjoined (5,8) move -> Linear Nimhoff: **fluctuations remain**, but there is also a hint of a new dashed P-line between the upper P-beams, so the conjecture needs some modification still…

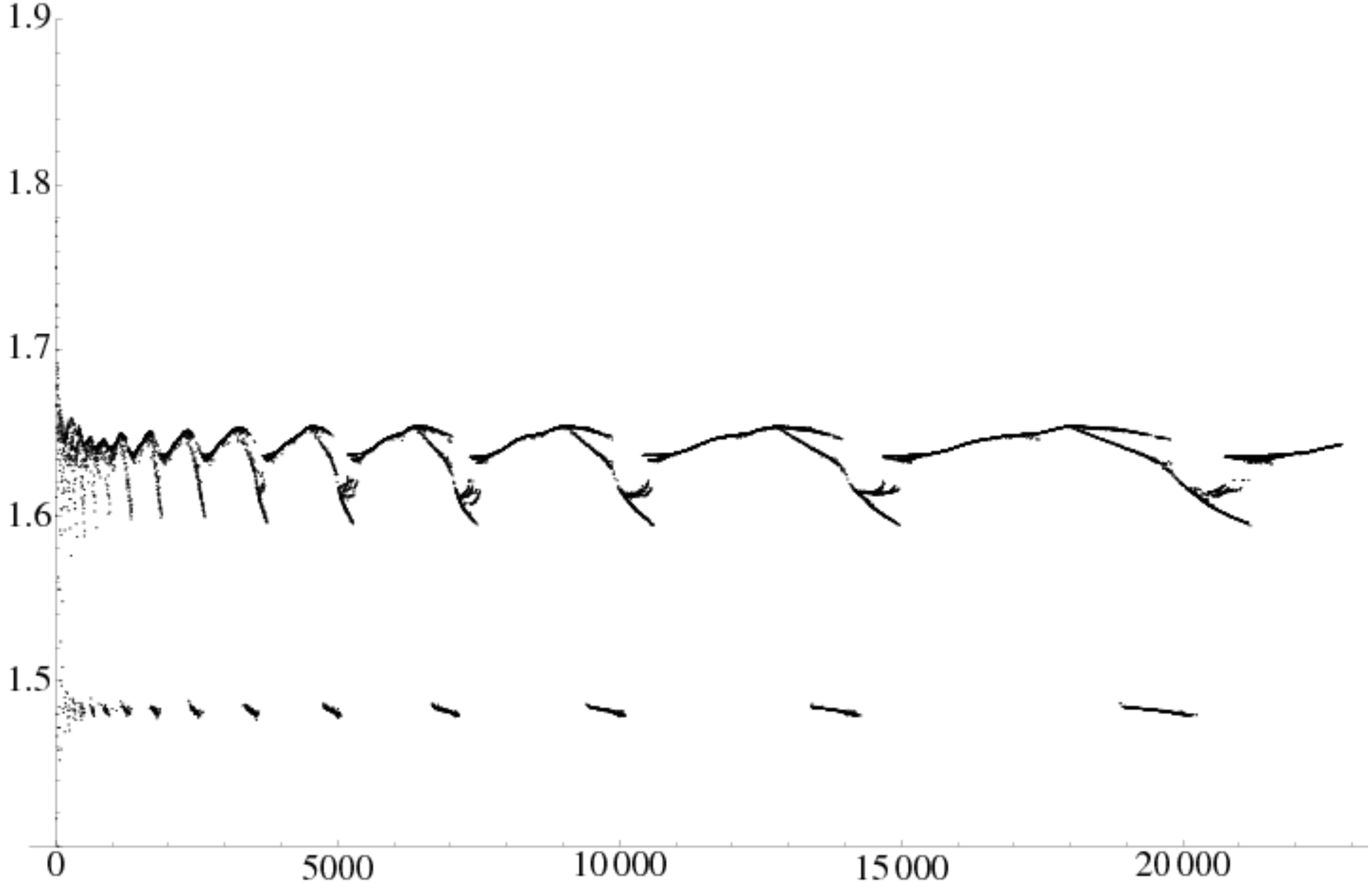


Equations for (p,q)-GDWN

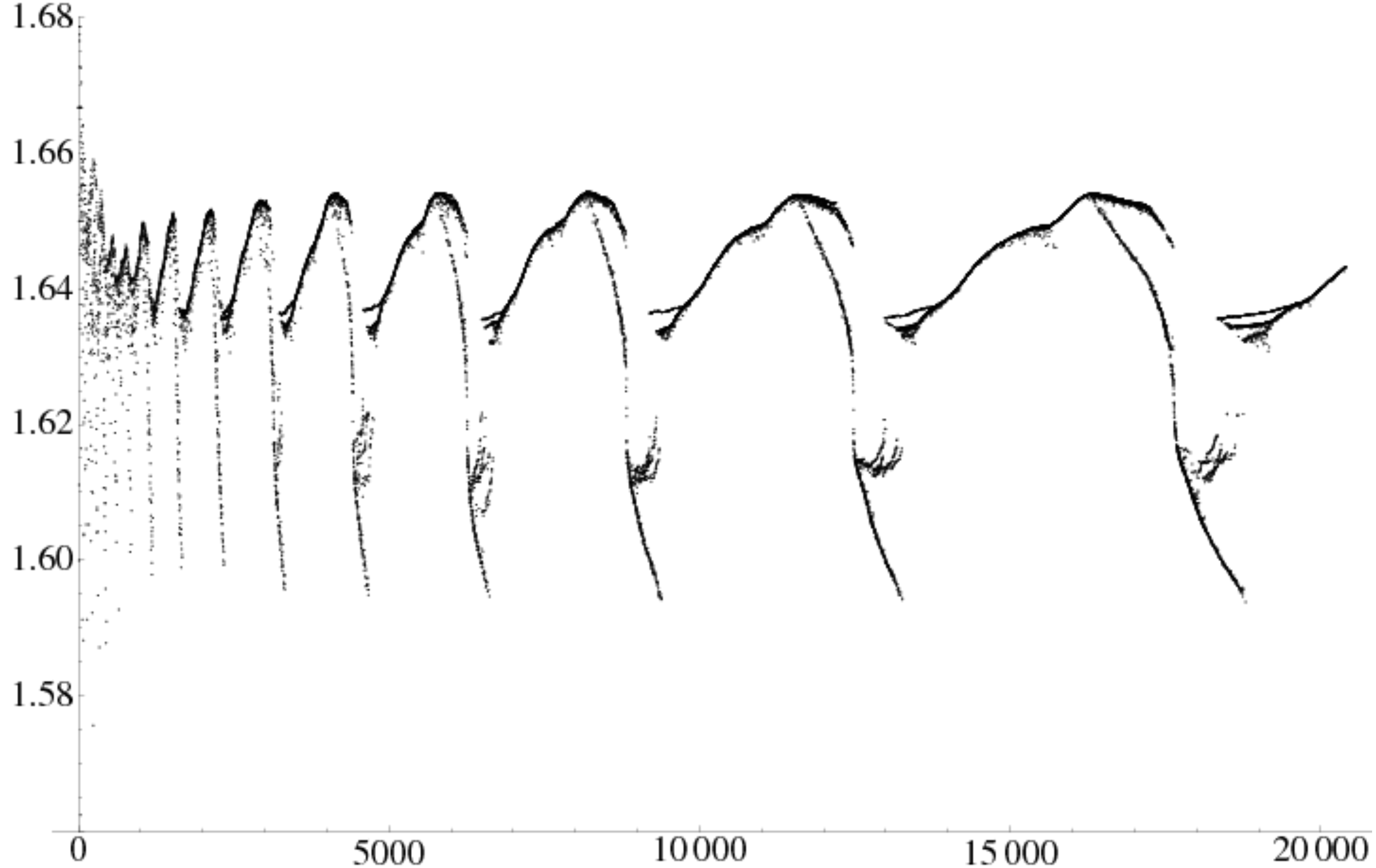
In general for (p,q)-GDWN, it is convenient to note that $m_1 = 1/m_4$, $m_2 = 1/m_3$, $m_1 = \lambda_1/\lambda_4$ and that $m_2 = \lambda_2/\lambda_3$. Thus, to compute the slopes, is suffices to solve the system

$$\begin{aligned}
1 &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4; \\
\lambda_1 &= p - q\lambda_1/\lambda_4; \\
1 &= \lambda_1\lambda_4/(\lambda_4 - \lambda_1) + \lambda_2\lambda_3/(\lambda_3 - \lambda_2); \\
1 &= \lambda_2\lambda_3/(q\lambda_2 - p\lambda_3) + \lambda_2\lambda_3/(q\lambda_3 - p\lambda_2) + \lambda_1\lambda_4/(q\lambda_4 - p\lambda_1).
\end{aligned}$$

Variations to the upper slopes of (4,6)-GDWN



The upper P-beam



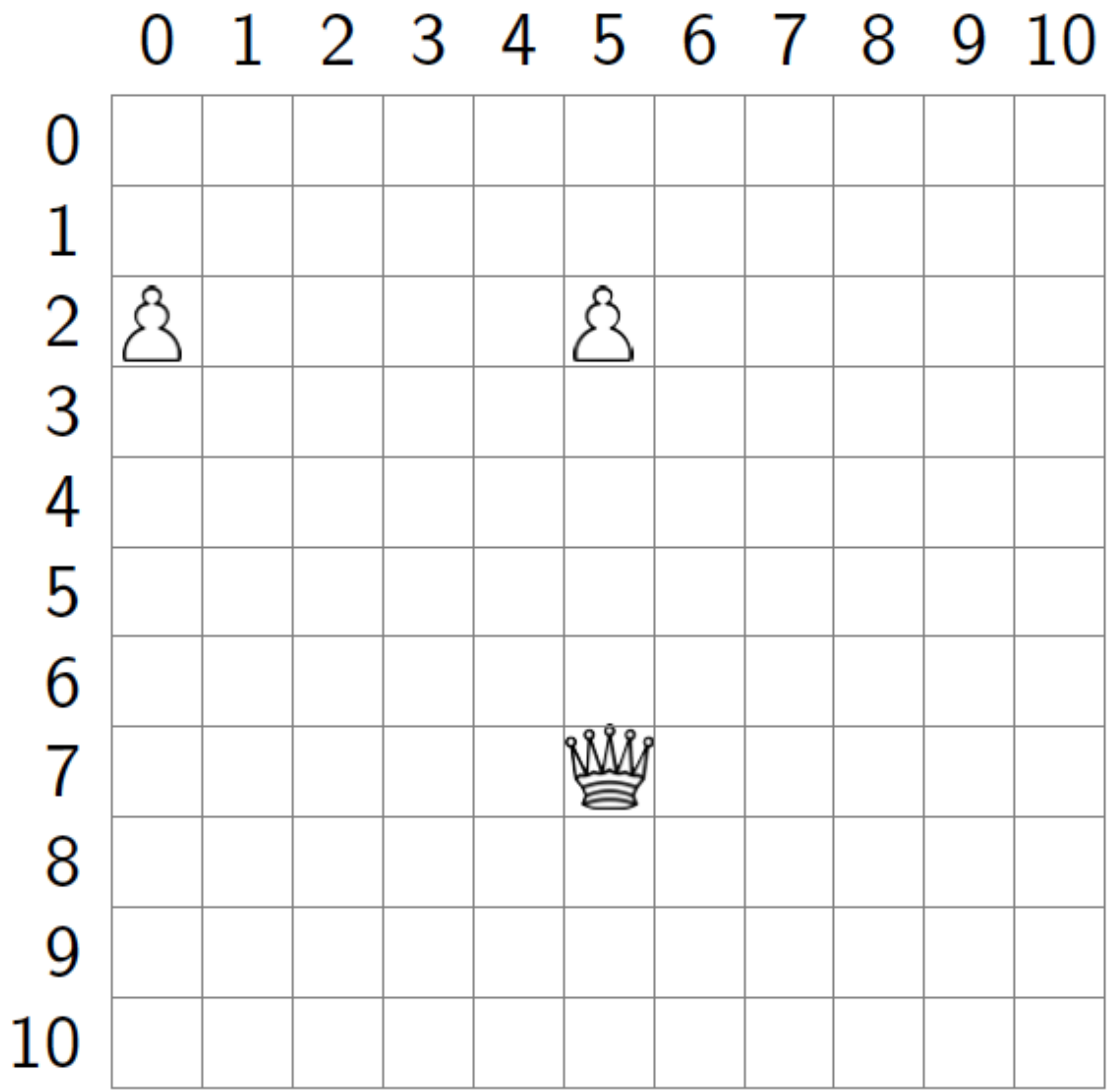
Embryonic development of a P-beam

A cellular automaton for blocking queen games

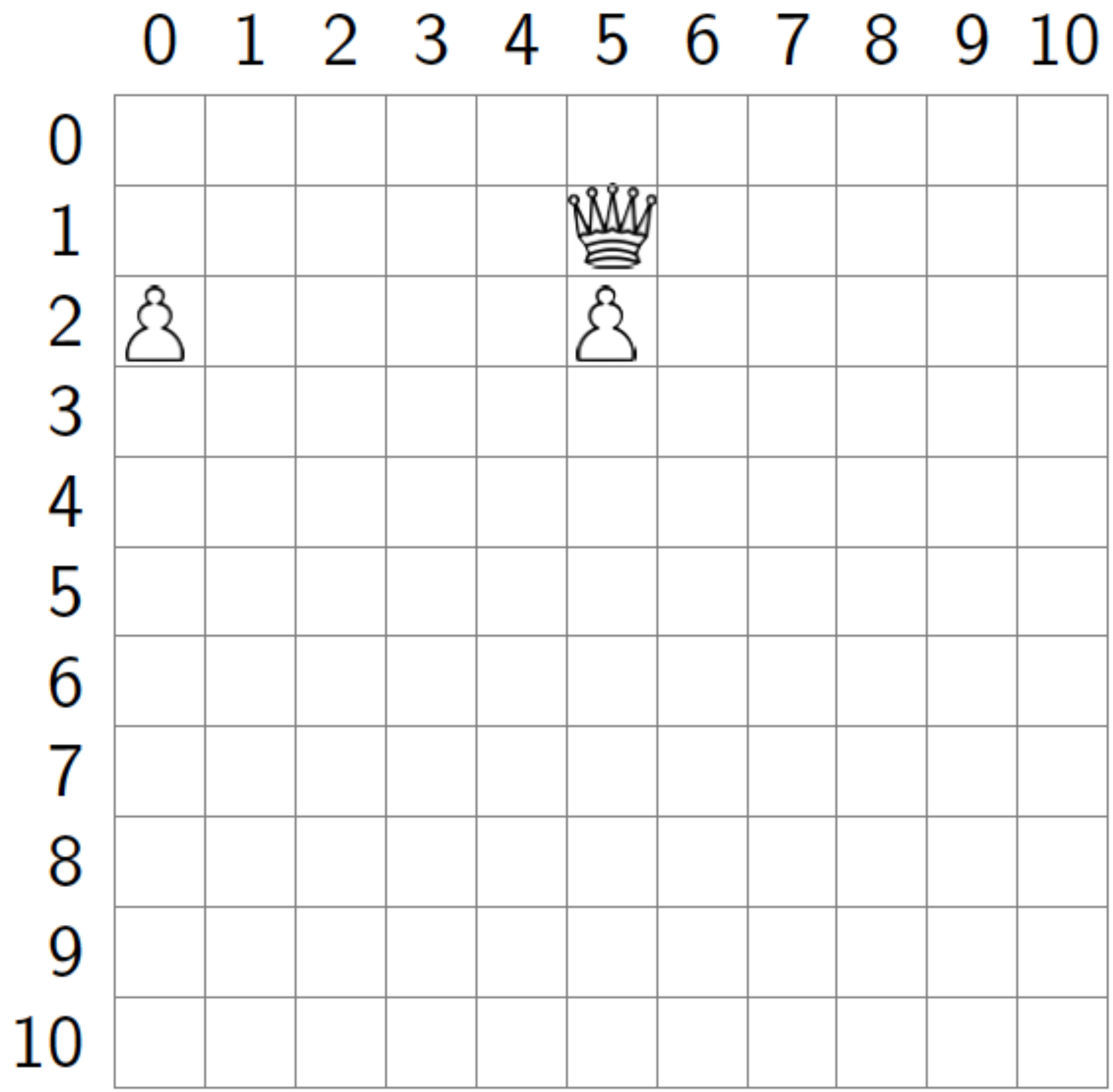
with Neary and Cook

…more structure to the game k-Blocking Wythoff Nim

Larsson 2011

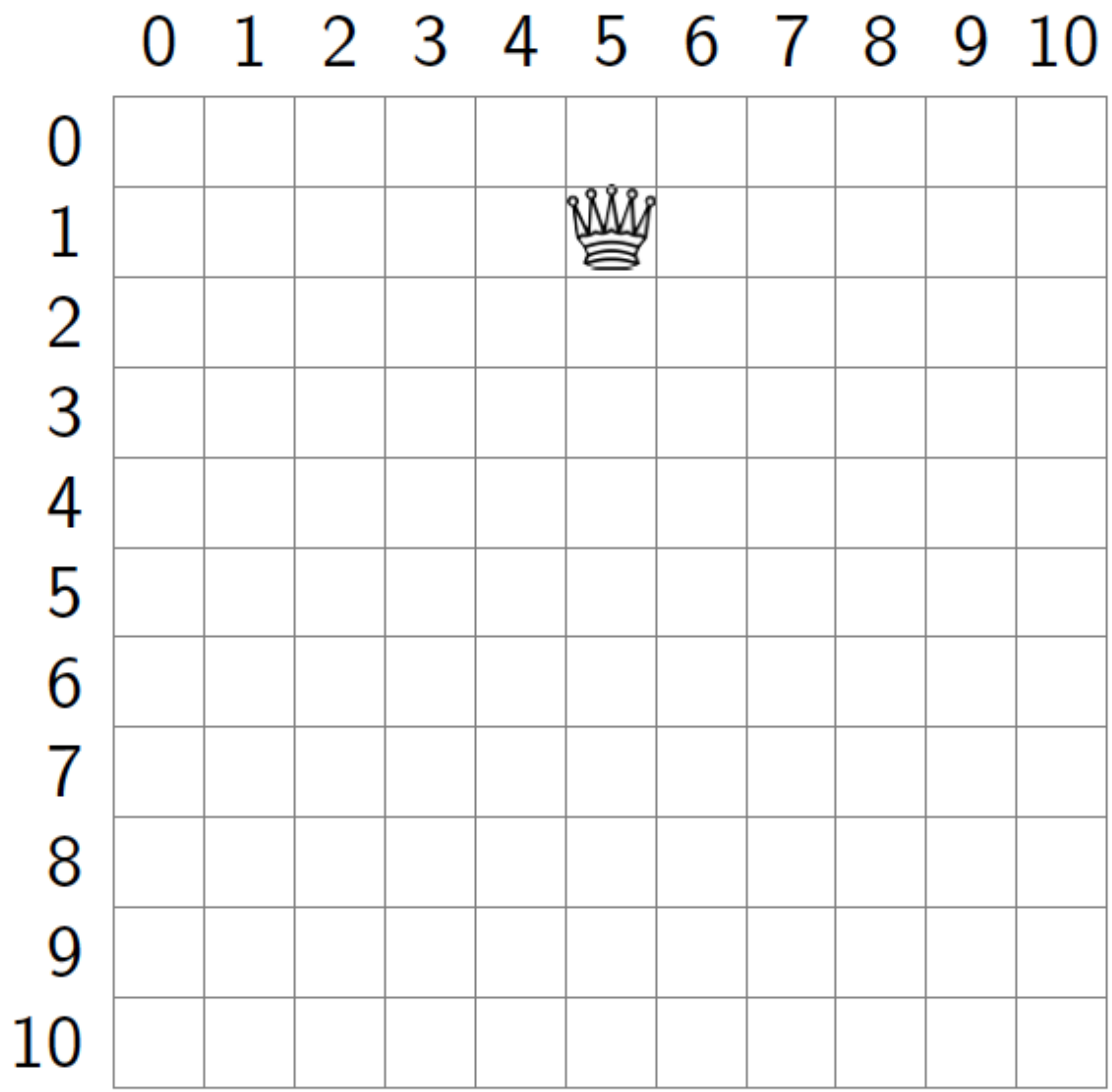


The terminal position is the upper left corner. Two of the queen's in total 17 move options have been blocked off

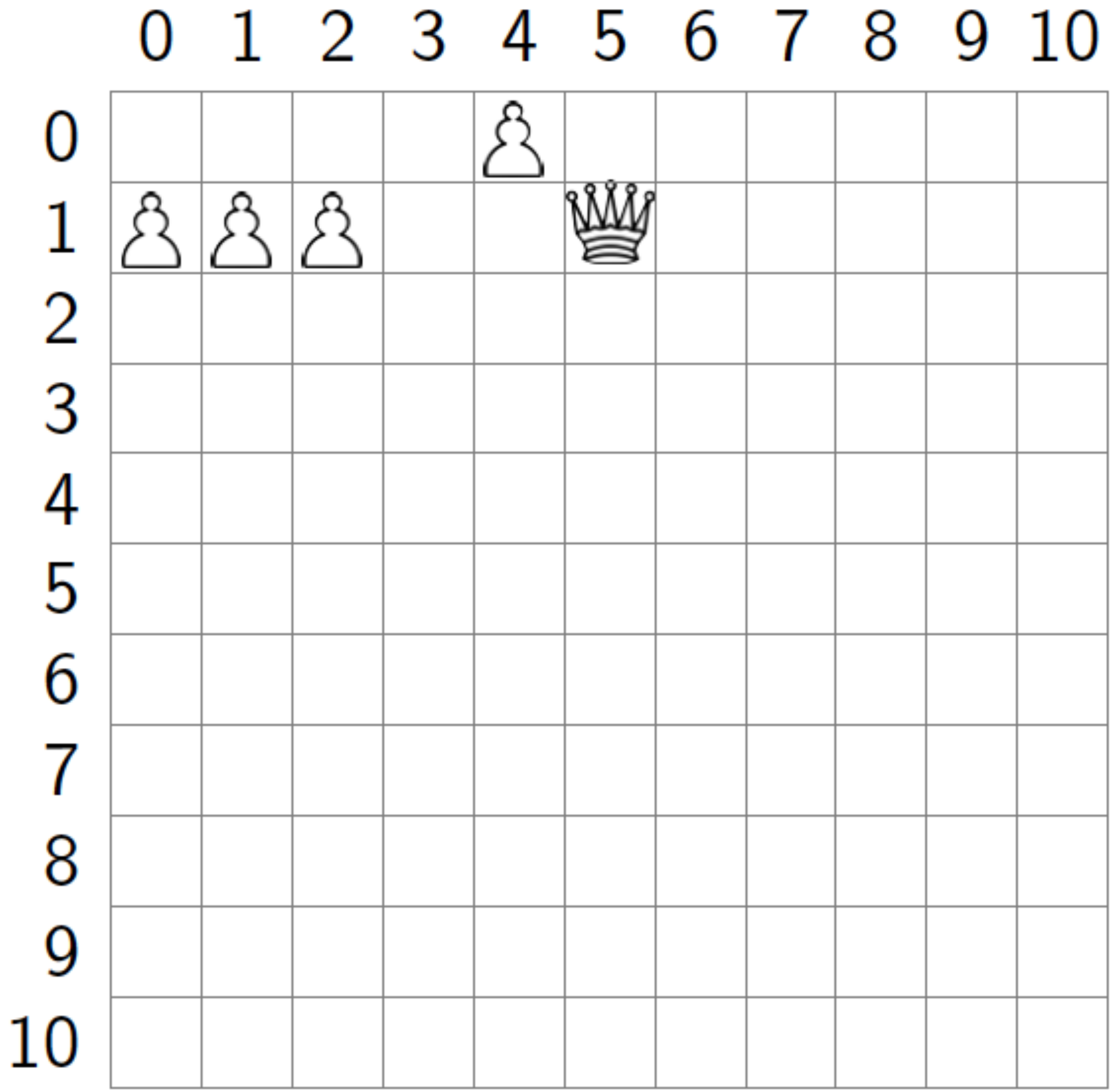


The queen can move to a non-blocked position, say $(5, 1)$.
The pawns get removed

k=5, the fifth move attempt cannot be blocked

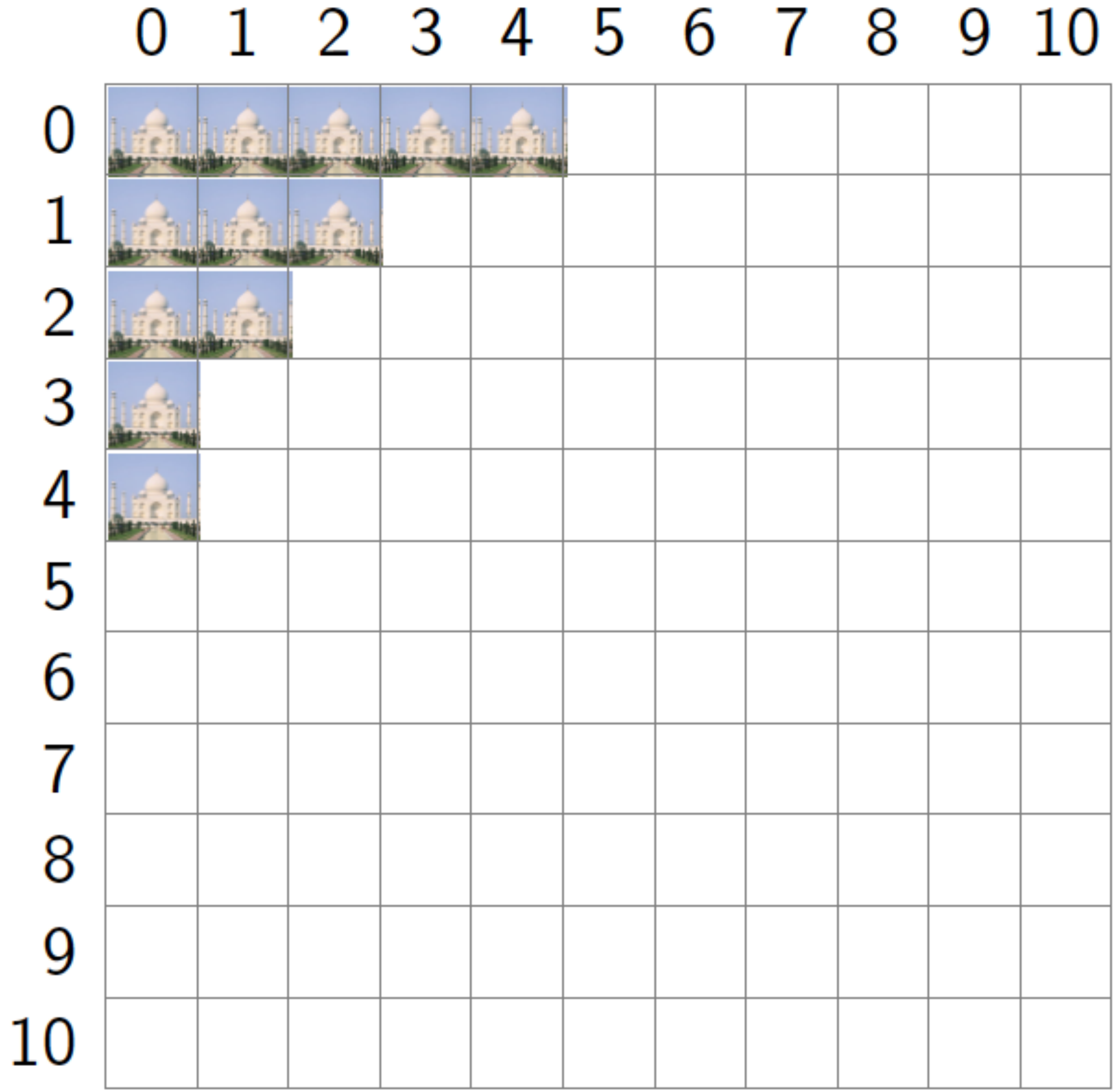


Suppose that the previous player is allowed to block at most four positions. Find your winning configuration of pawns!

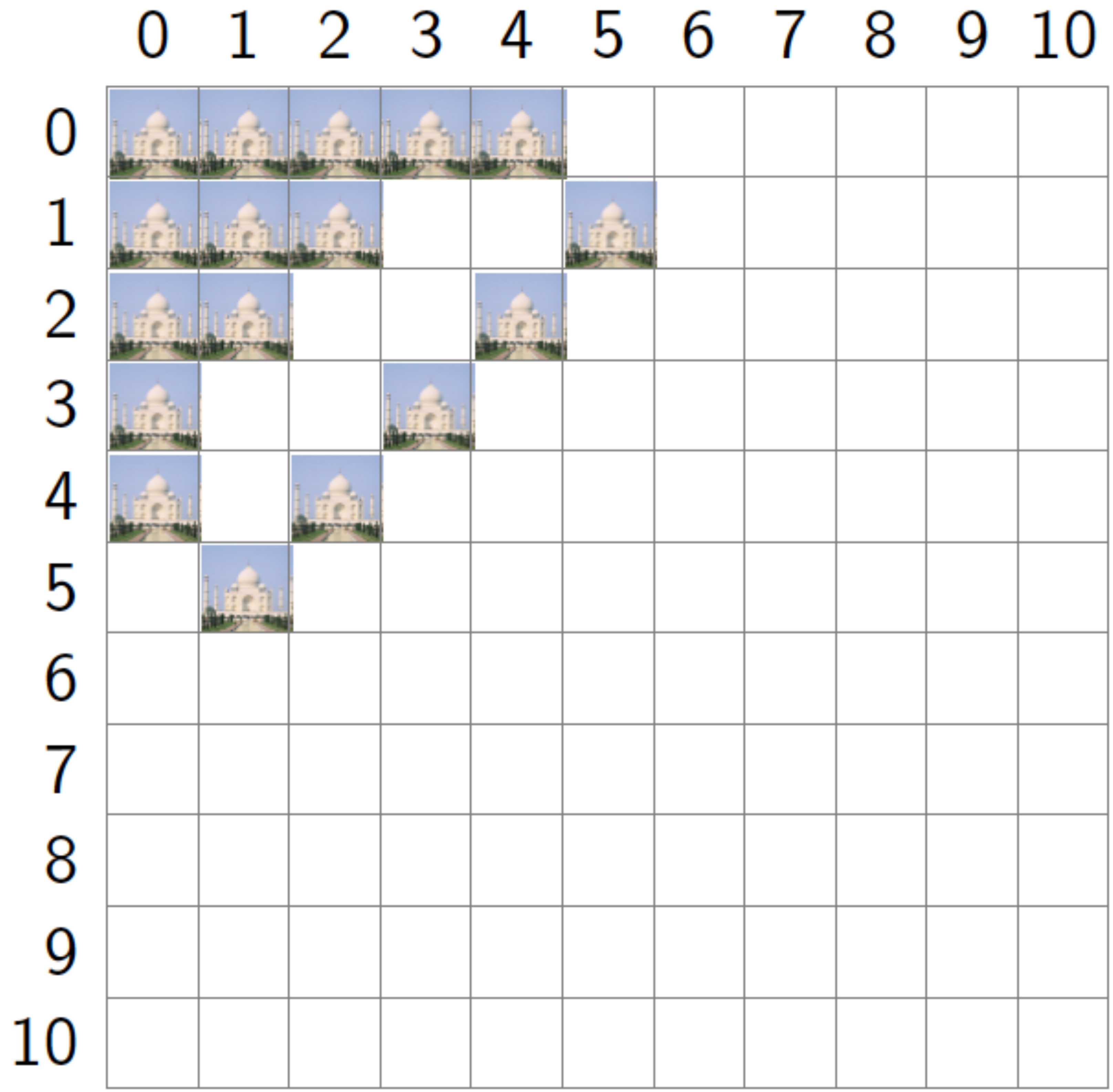


Correct!

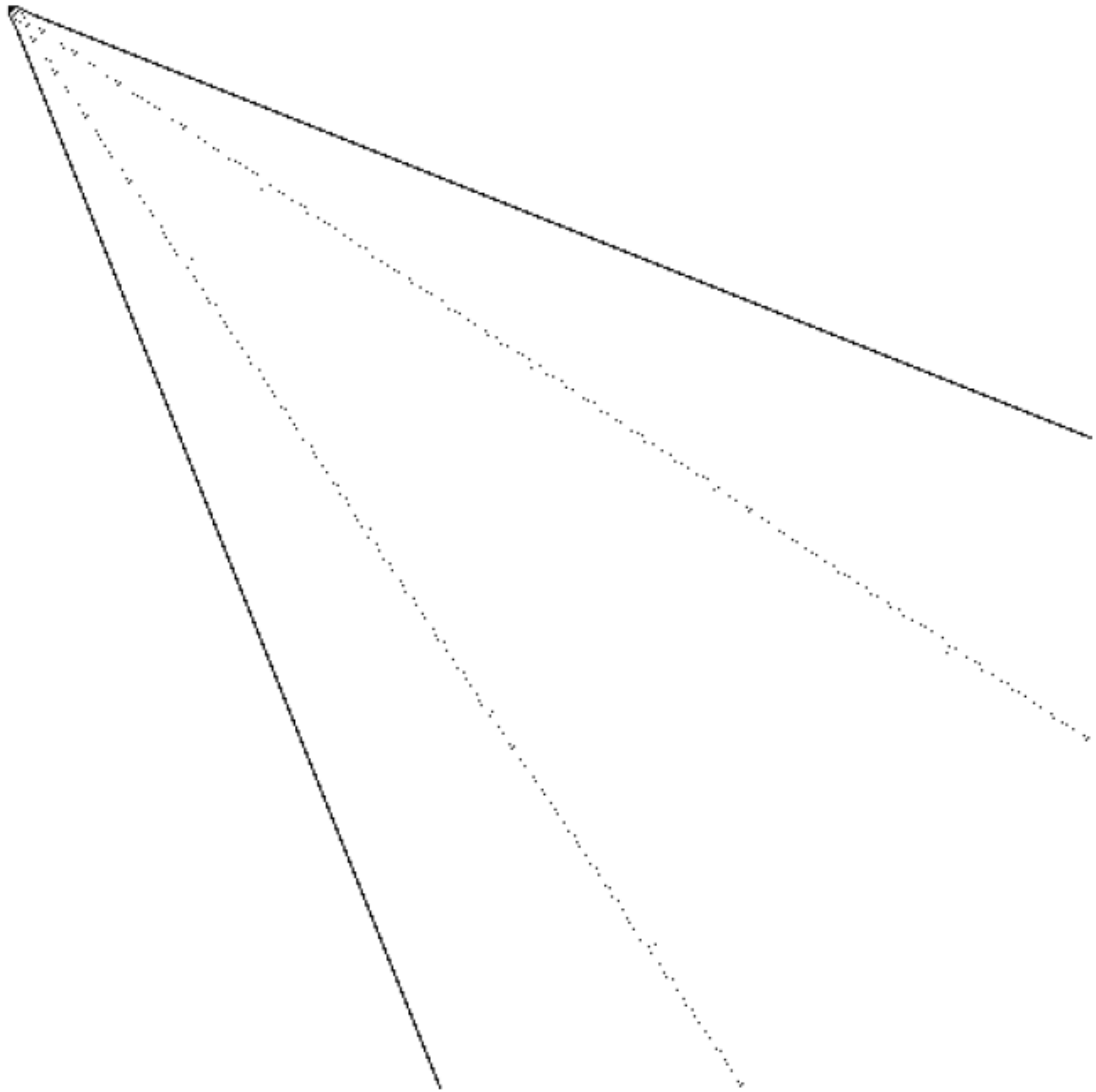
Well done!



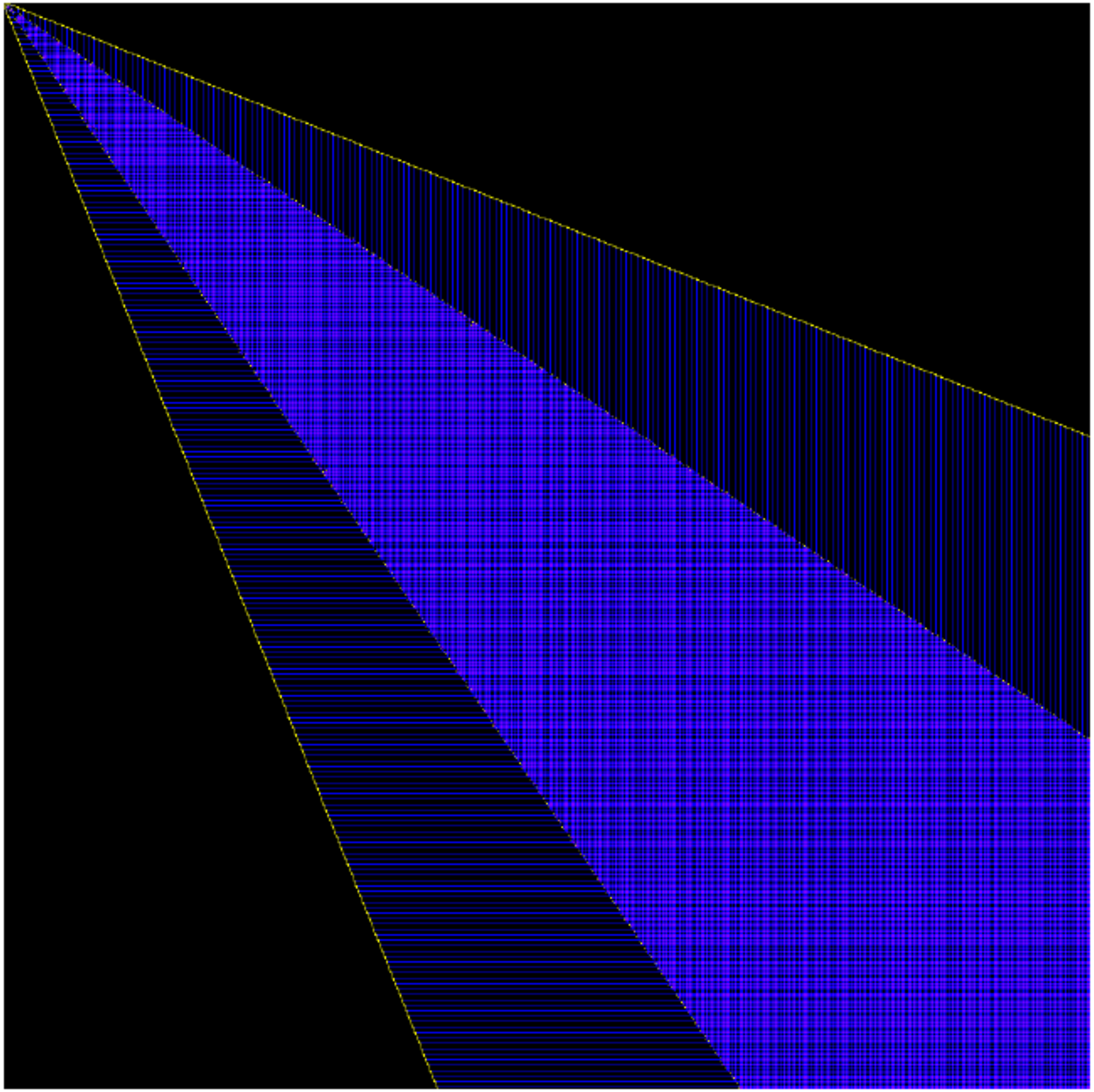
All terminal palaces for $k = 5$. From each such palace the previous player can block off all moves



The terminal positions and the next level of palace positions, those who see only terminal palaces.



The palaces for $k = 5$, 1000×1000 positions



The palace numbers for $k = 5$ reveal more structure

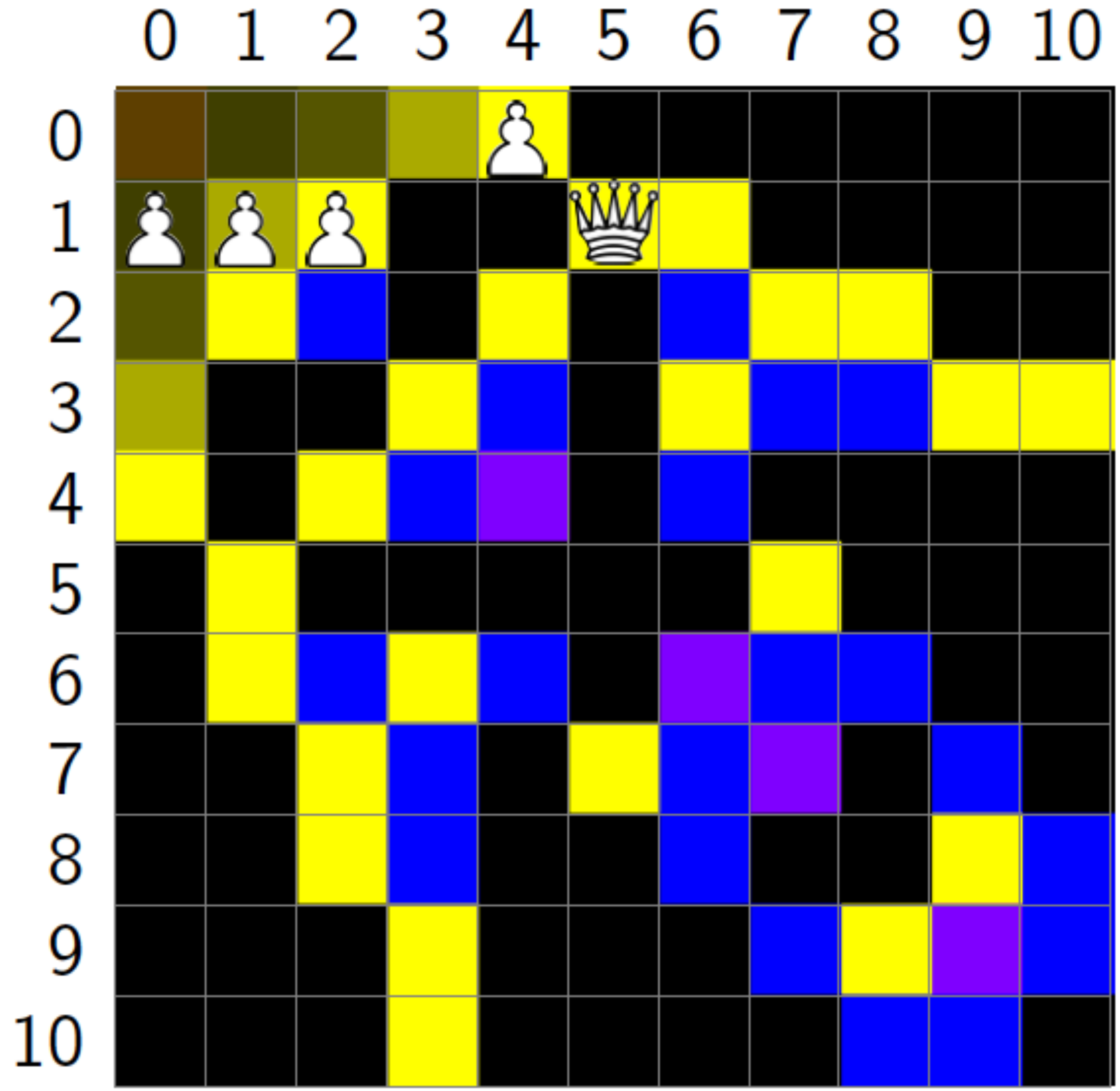
A color coding of palace numbers. The colors represent the number of winning moves available (ignoring blocking) from each position for the blocking parameter $k = 5$.

P-positions (the previous player wins)

0: Dark Brown, 1: Dark Olive, 2: Olive, 3: Light Olive, 4: Yellow,

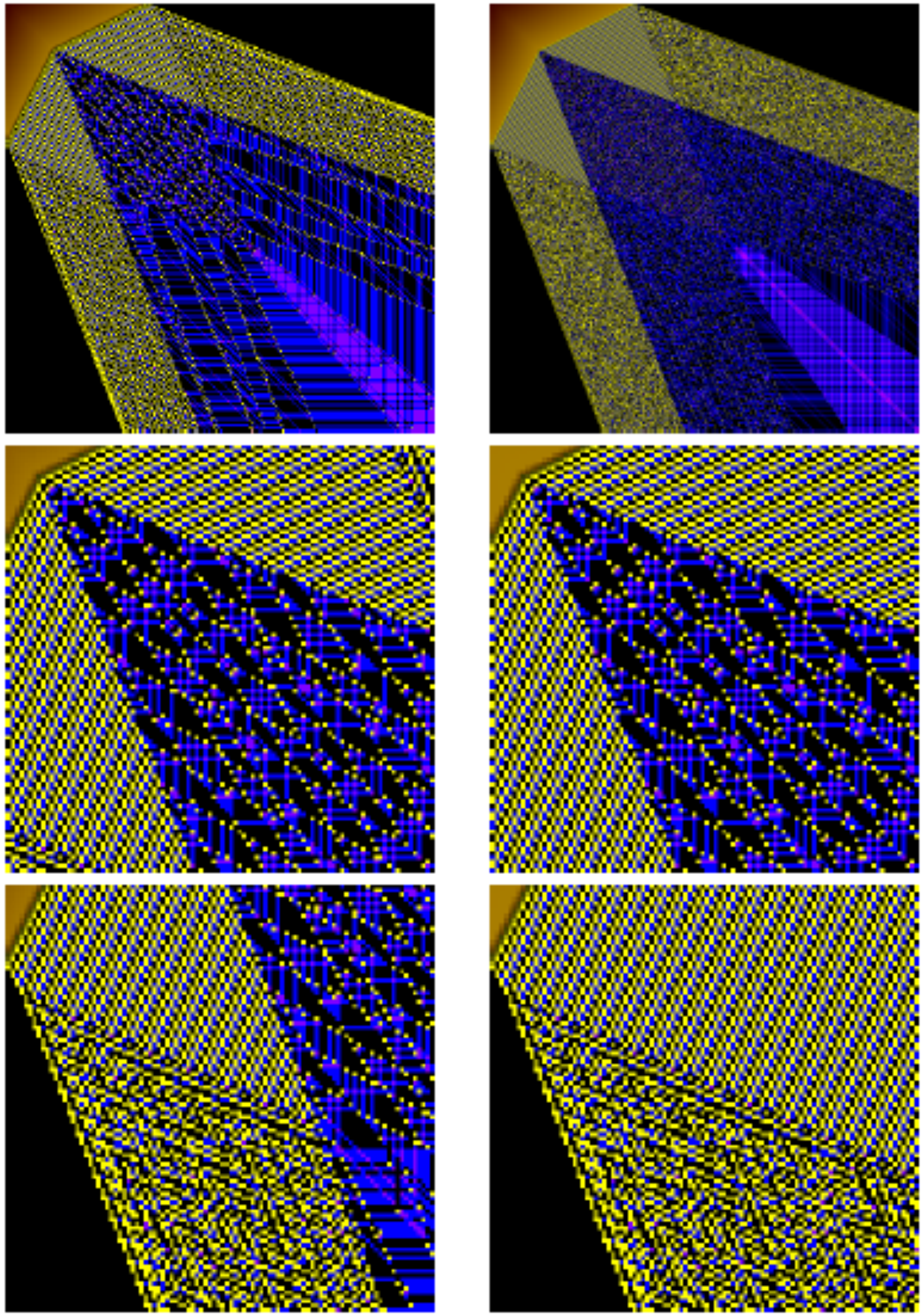
N-positions (the next player wins)

5: Black, 6: Blue, 7: Indigo.



This color coding shows that you will win, since the opponent will only be able to move to black palace numbers!

k=100 k=1000



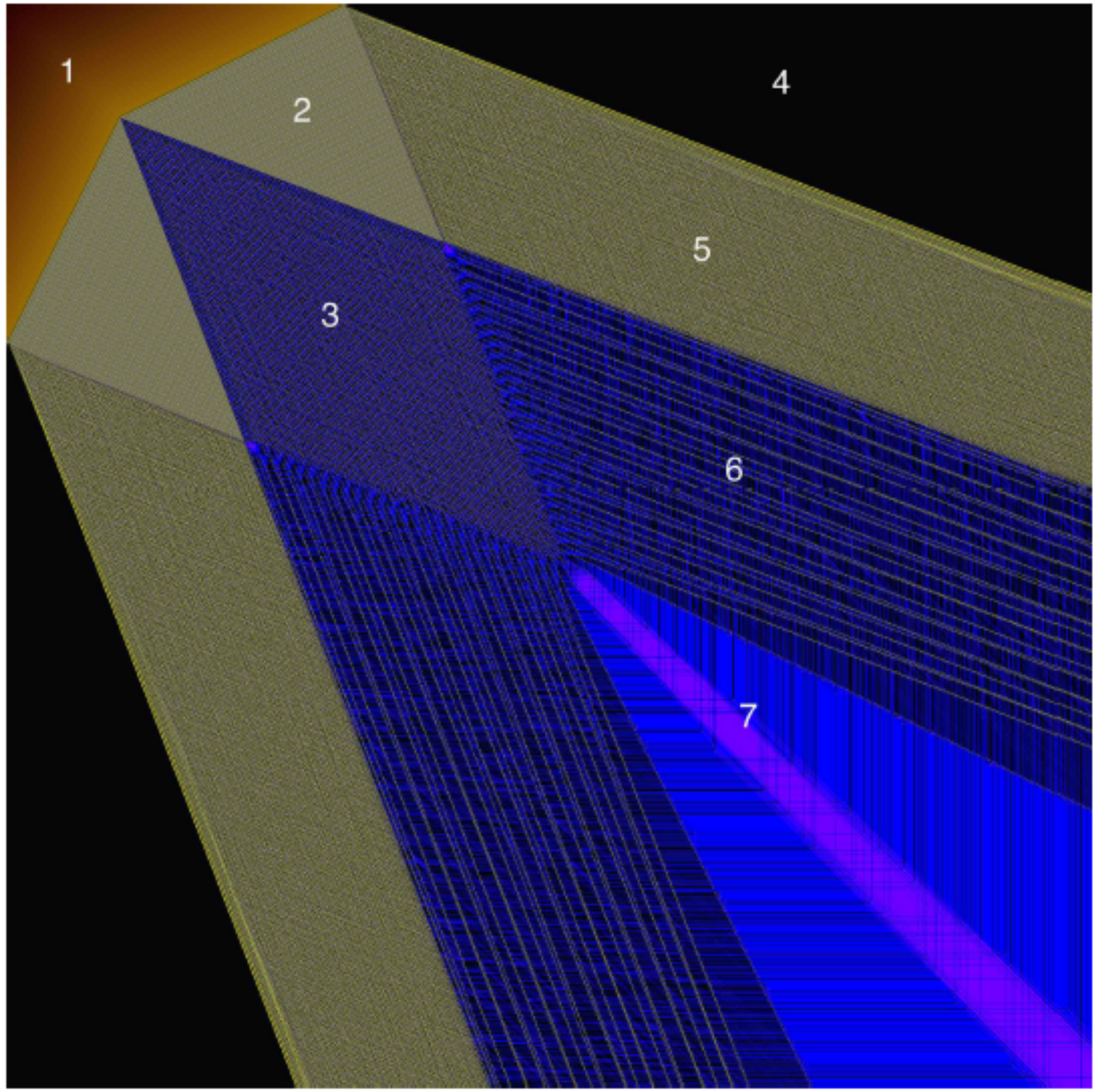
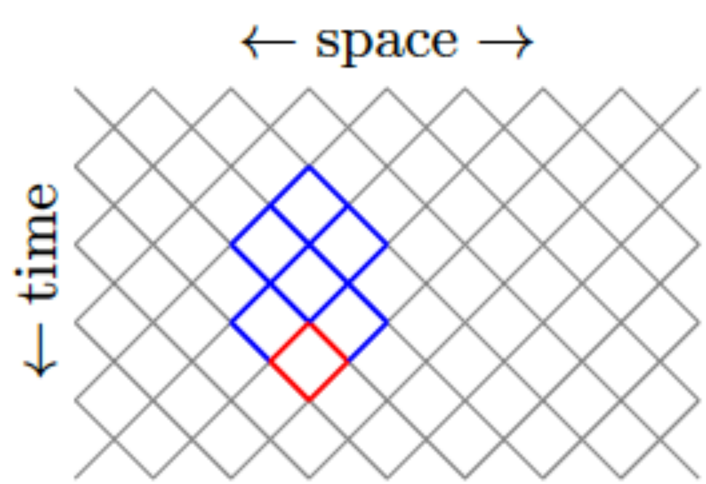
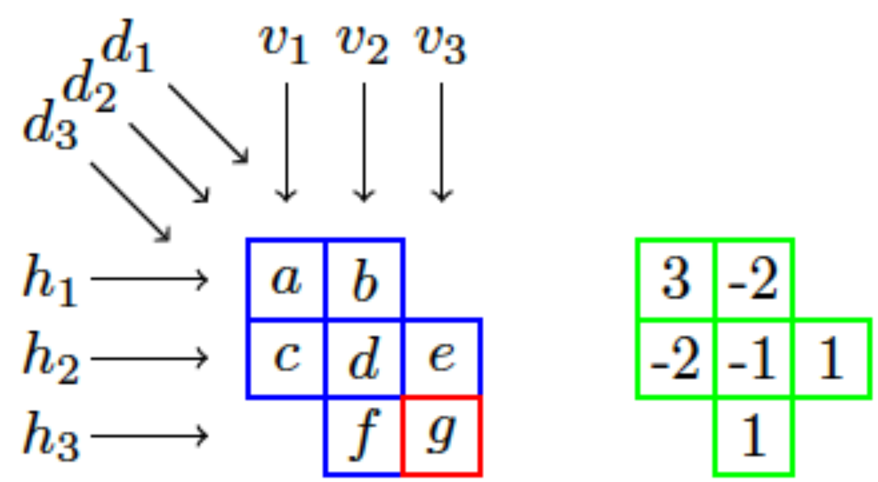


Fig. 4. The seven self-organized (large k) regions for blocking queen games: (1) the hood (2) an épaulette (3) the fabric (4) the outer space (5) an arm (6) the warp (7) the inner sector. Some triples of regions define junctions and some pairs define borders: i.e. (1,2,3) the nose (1,2,5) a shoulder (3,5,6) an armpit (3,6,7) the prism (1,2) a casing (2,5) a hem (2,3) a rift (3,6) a fray (4,5) a skin.

The cellular automaton for k-Blocking Wythoff Nim





$$a + e + f \;=\; \sum_{i=1}^{3} v_i \;+\; \sum_{i=1}^{3} d_i \;+\; \sum_{i=1}^{3} h_i \;-3k =\; b + c + g$$

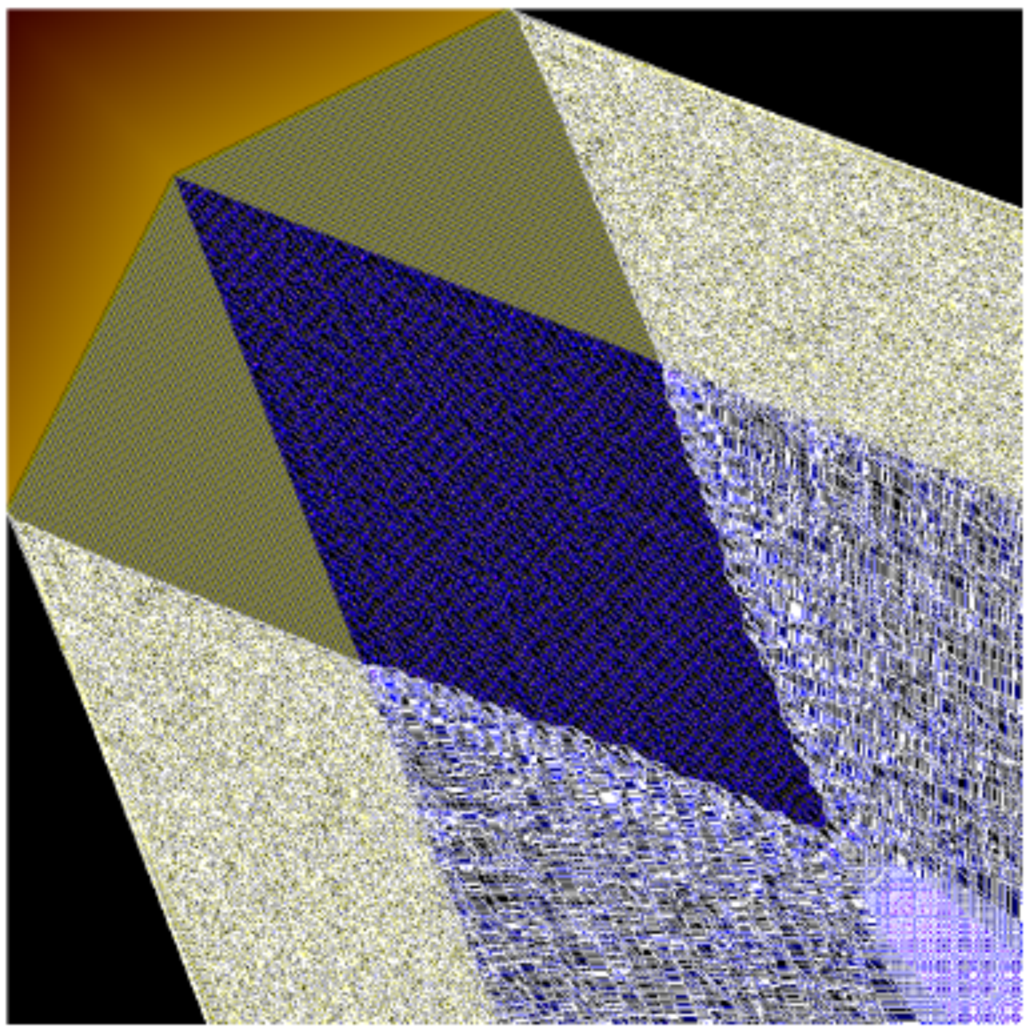
$$g = a - b - c + e + f + p.$$

p is the sum of green cells that
correspond to blue cell’s with a negative value

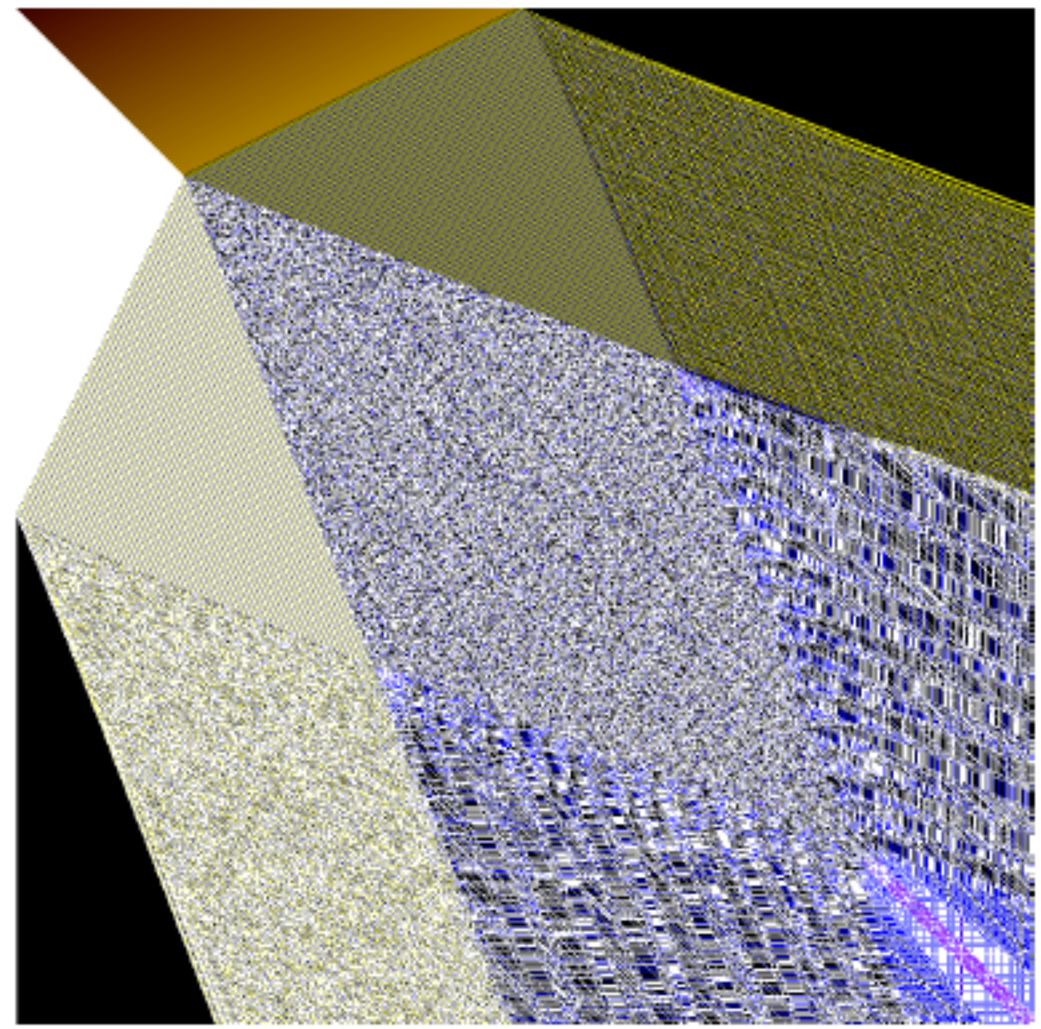
Theorem 1. *The k-Blocking Wythoff Nim position (x, y) is a P-position if and only if the CA given in Figure 2 gives a negative value at that position, when the CA is started from an initial condition defined by*

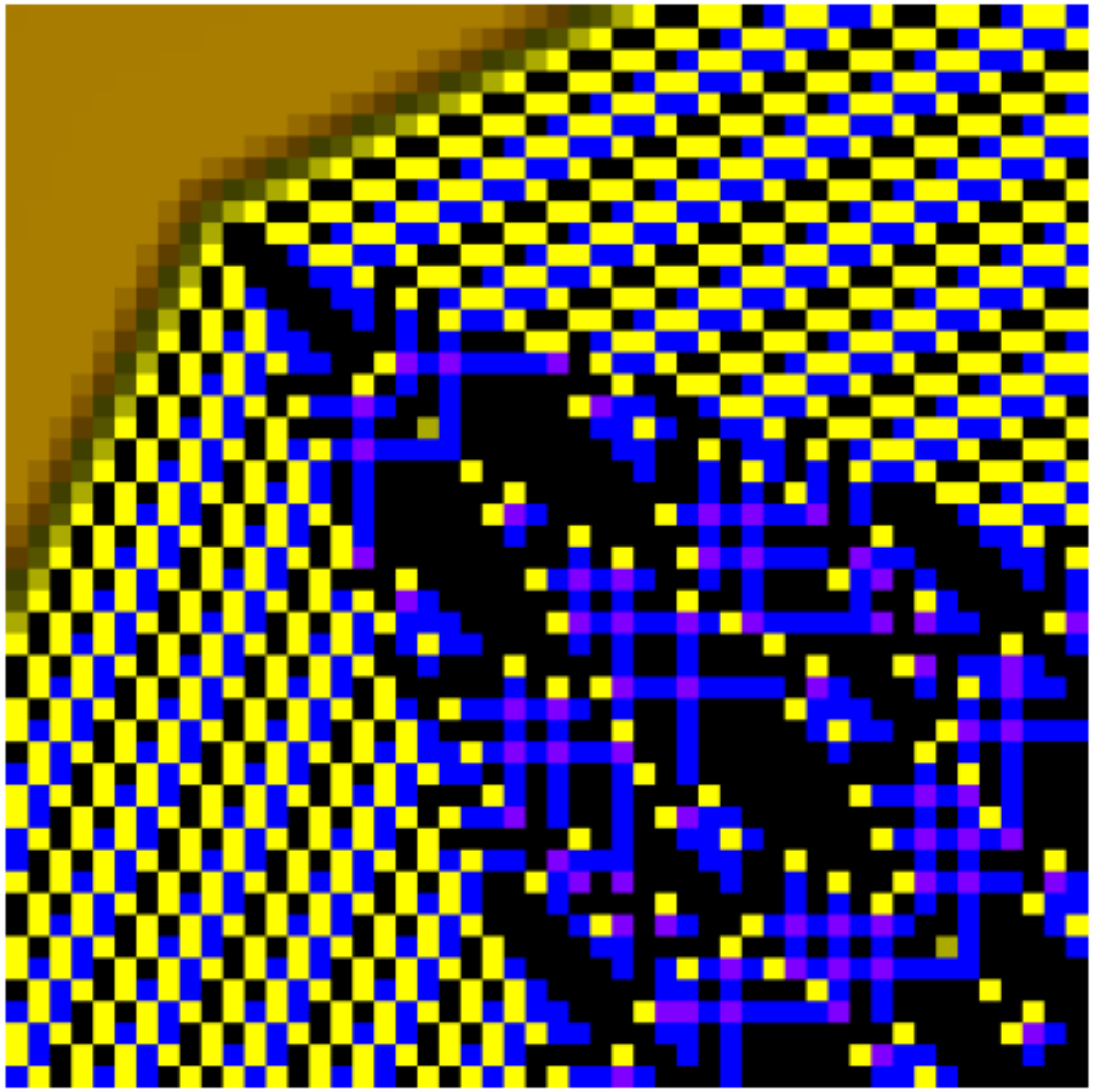
$$CA(x, y) = \begin{cases} k & x < 0 \text{ and } y < 0 \\ 0 & x < 0 \text{ and } y \geqslant 0 \\ 0 & x \geqslant 0 \text{ and } y < 0 \end{cases}$$

Difference between k=497 and k=500

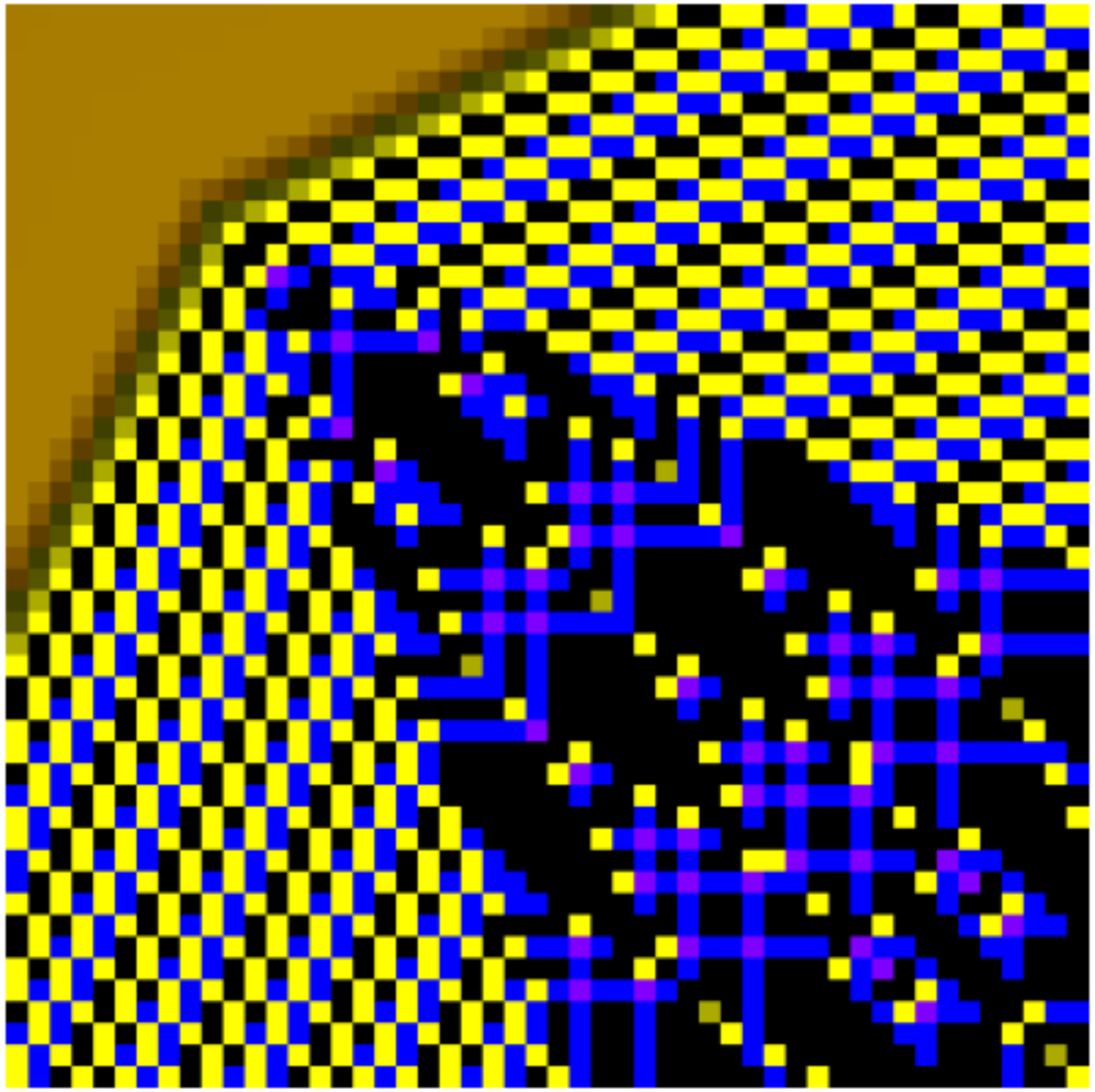


Difference between k=499 and k=500

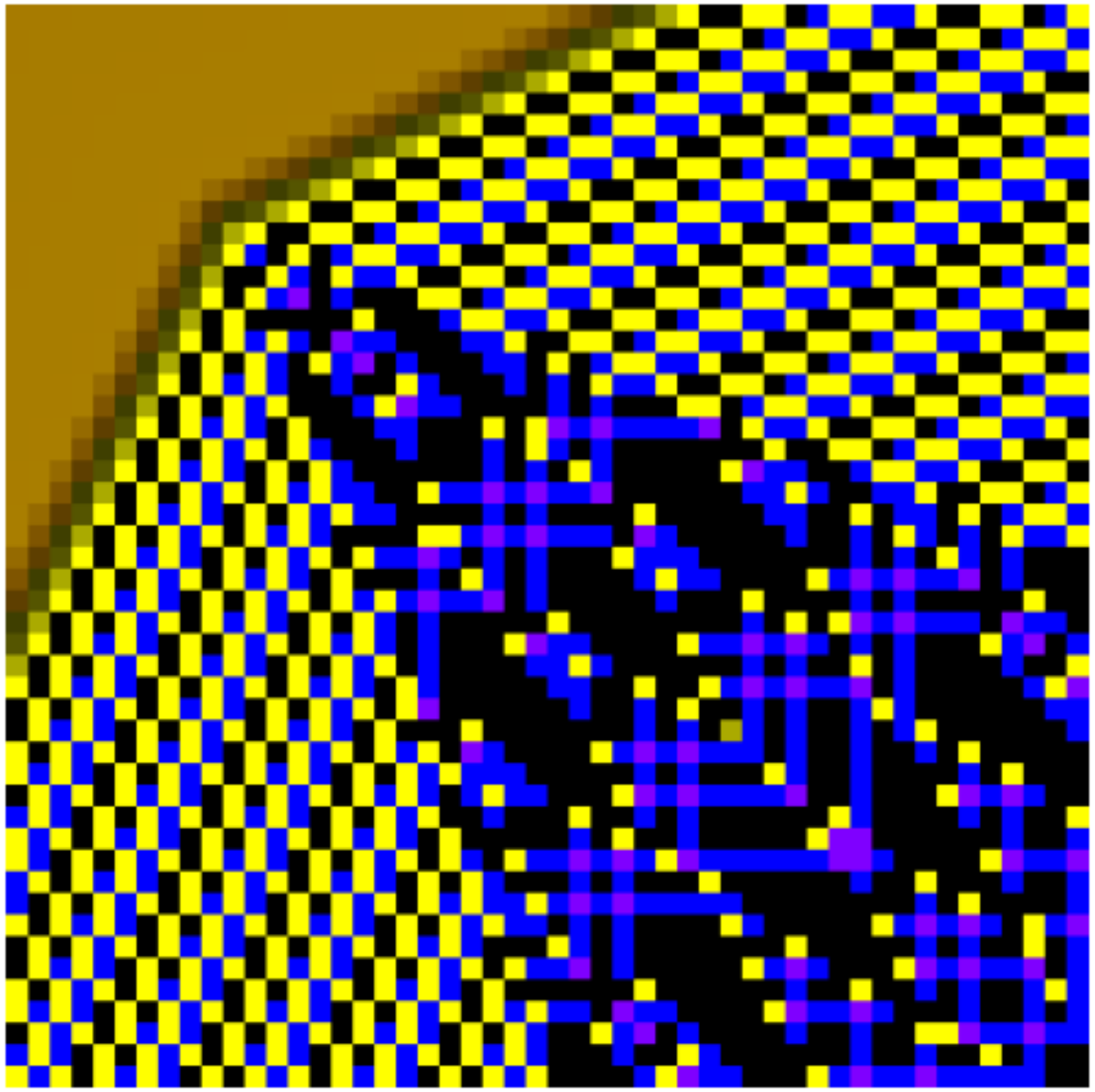




There are only three possible patterns of the ‘fabric’: 0

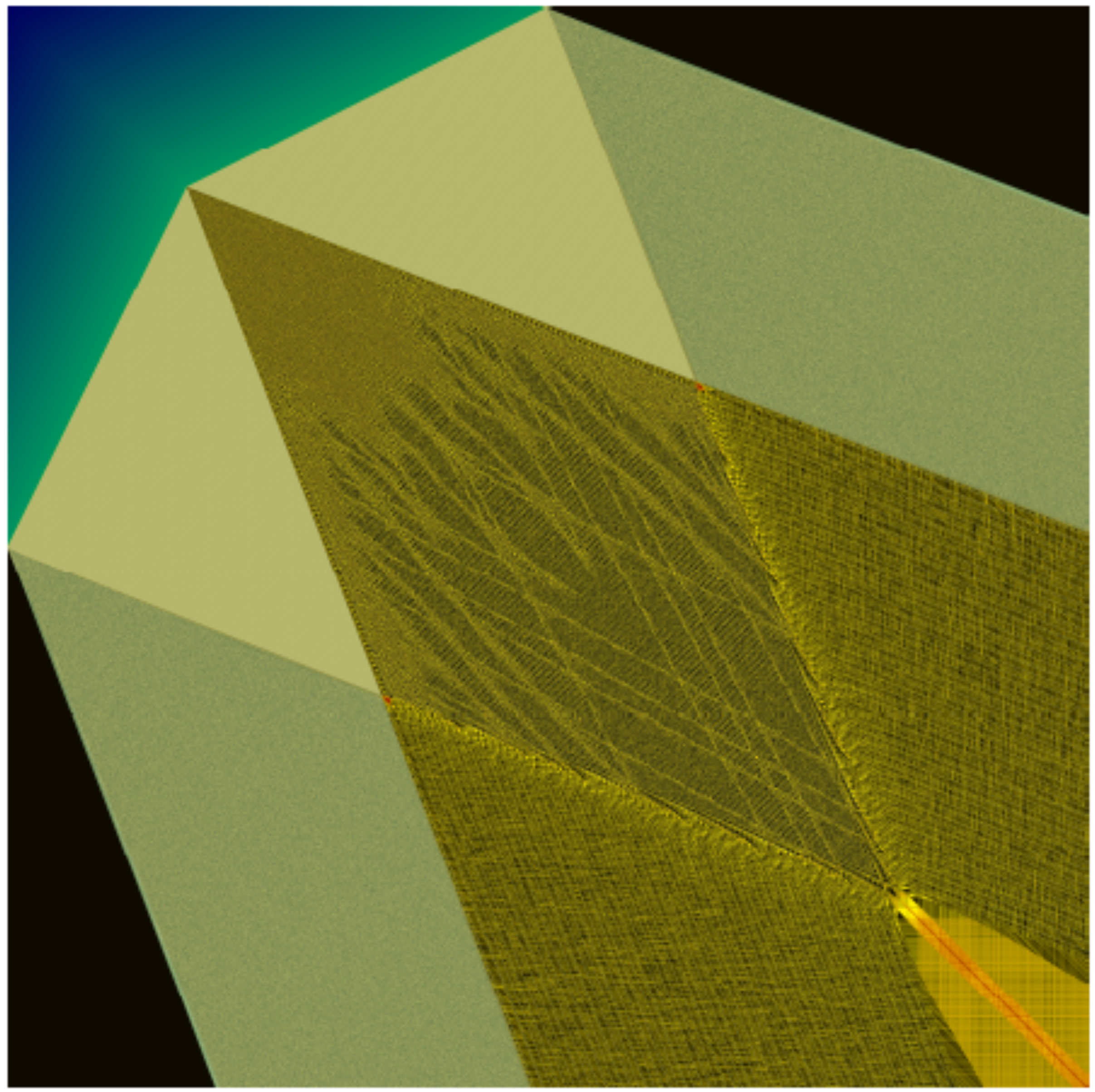


There are only three possible patterns of the ‘fabric’: 1



There are only three possible patterns of the ‘fabric’: 2

A different color map highlights some interesting behavior



Meta-gliders in the fabric of game 3999

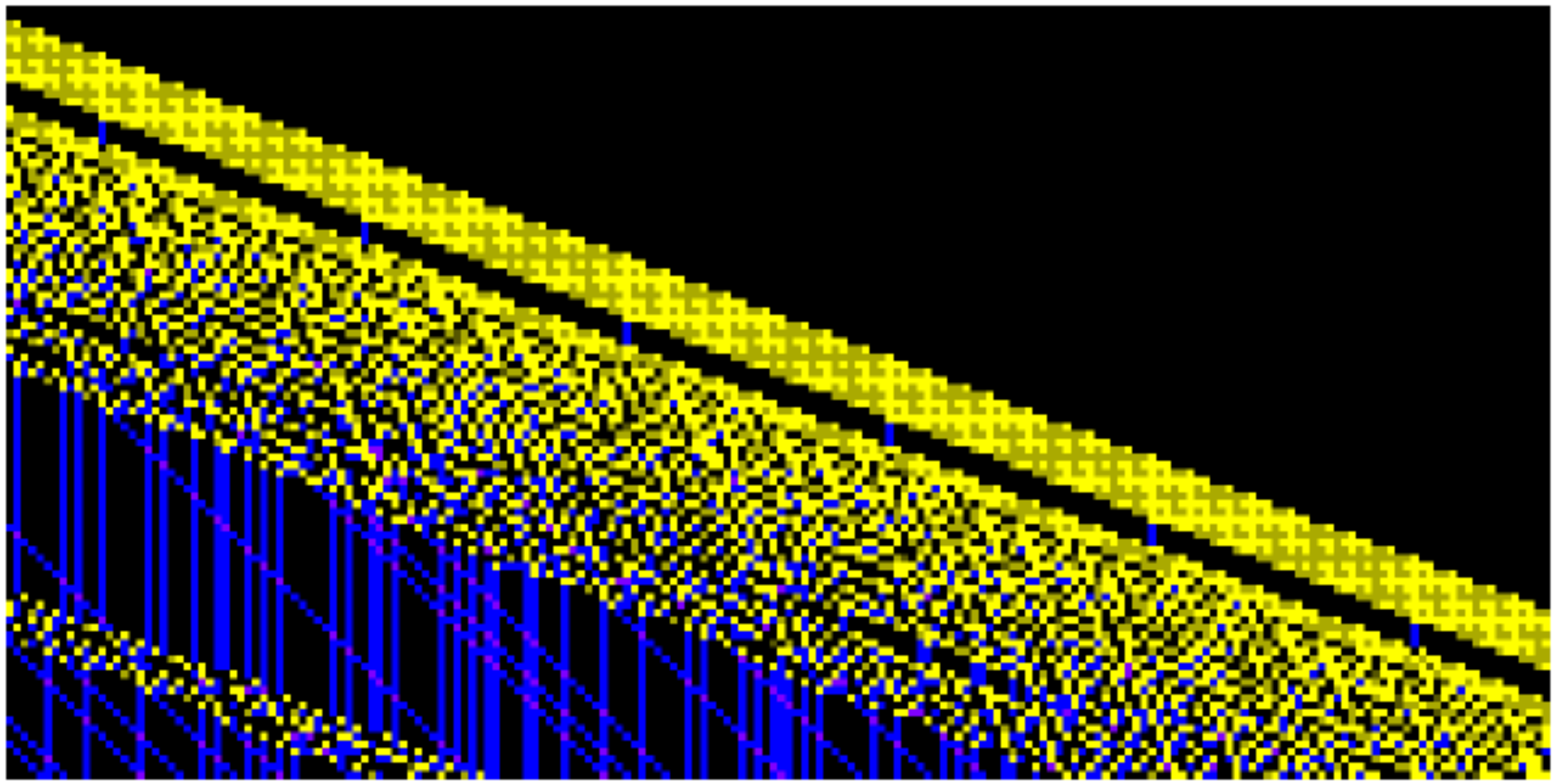
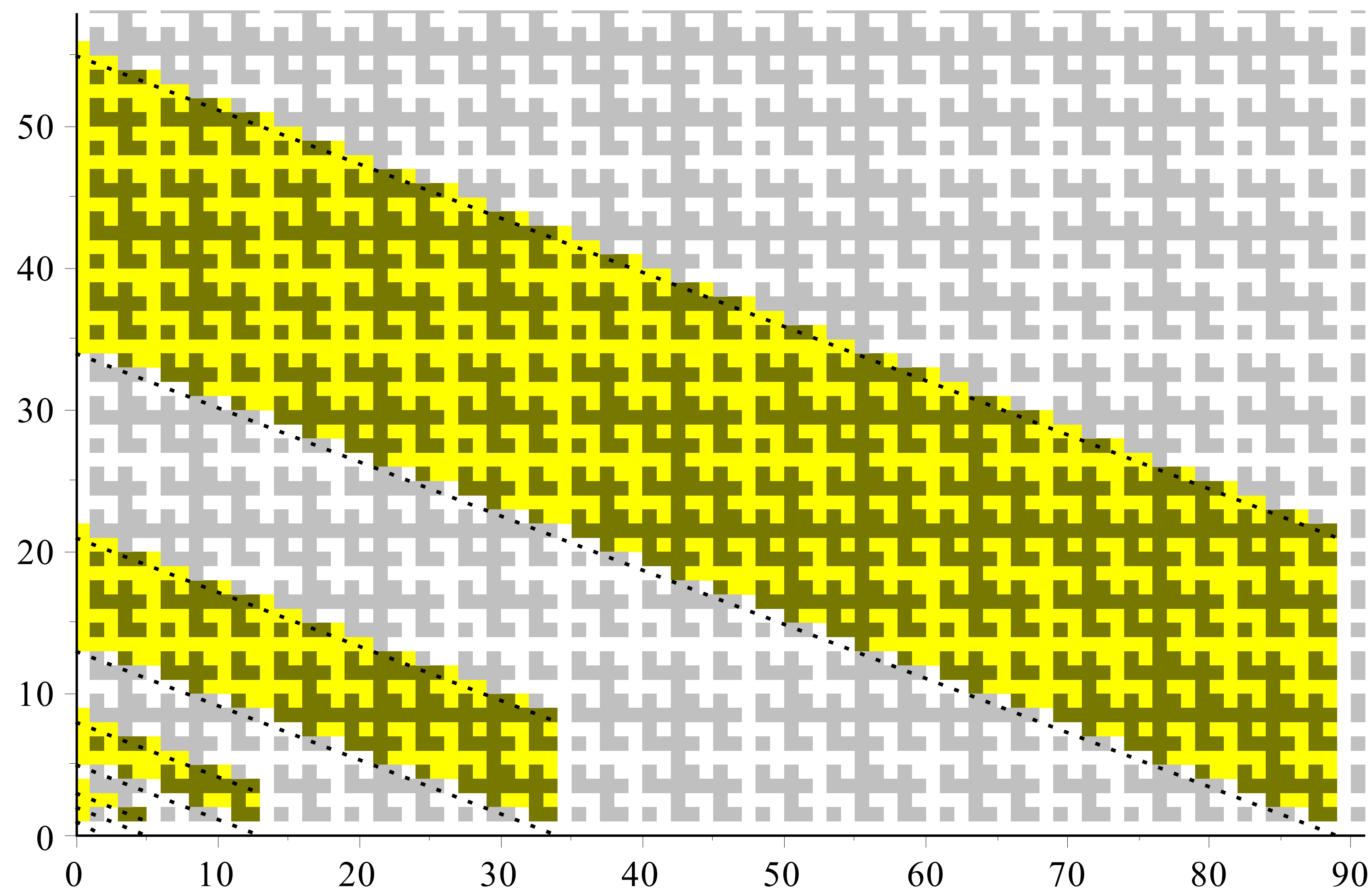


Fig. 9. Skin pattern. The diagonal black stripe crossing the picture from the upper left to the lower right separates the skin (solid yellow/olive stripe) above it from the arm below it. A blue vertical stripe is emitted by the skin once per period, in this case once every 34 columns. The period is always a Fibonacci number. This picture shows the case $k = 100$, in the region $(10600 \pm 100, 4050 \pm 50)$.

Kari and Szabados 2015

$$s(i,j) = \lfloor \phi(i+j) \rfloor - \lfloor \phi(i) \rfloor - \lfloor \phi(j) \rfloor$$



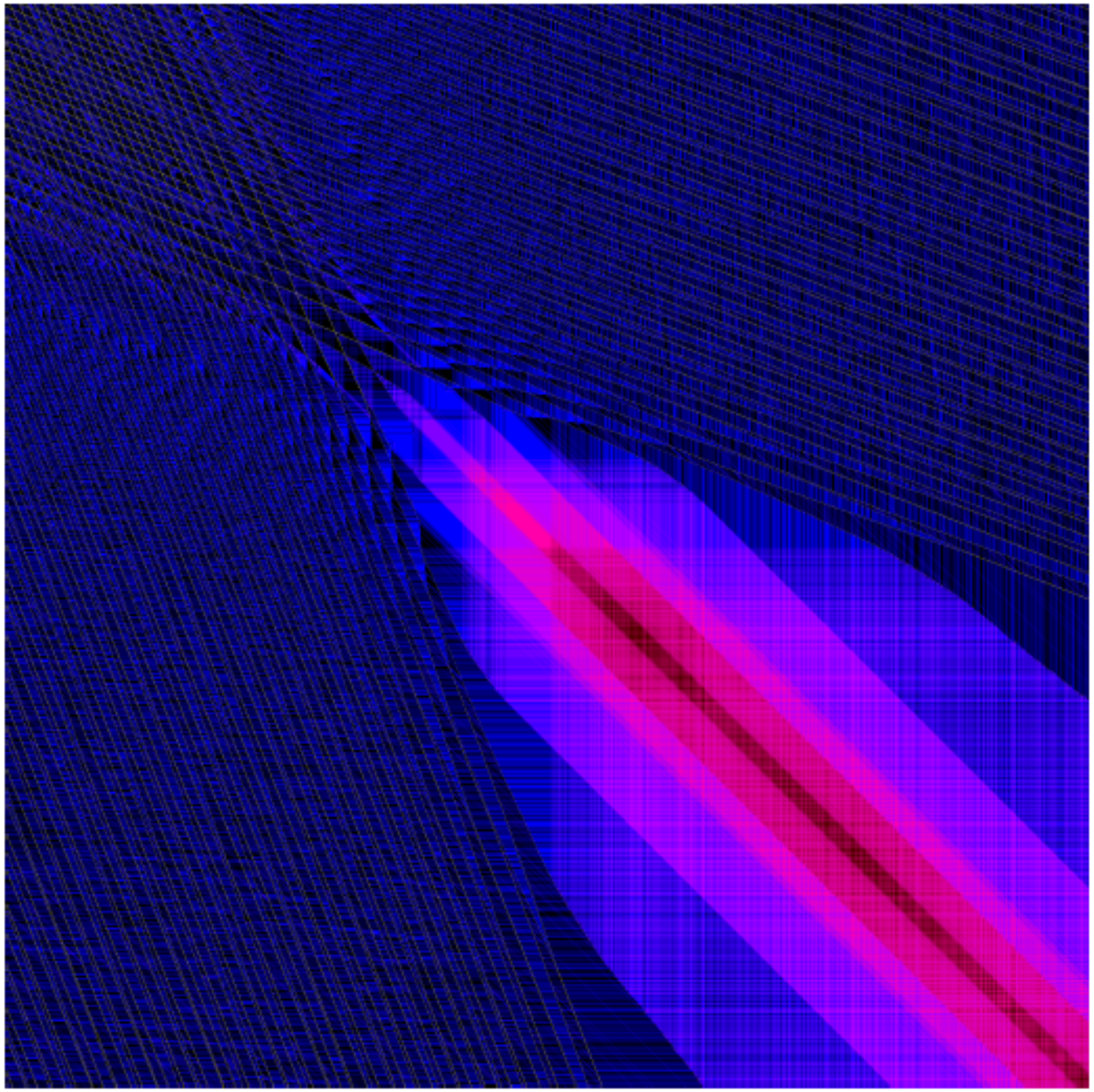


Fig. 7. Game 40003: A prism and its light beam. The slightly higher palace numbers near the main diagonal can give the impression of a beam of light being emitted by the prism. When the two fraying edges of the fabric meet at the prism, such higher palace numbers are often produced. If viewing this document electronically, zoom in for detail.