INVARIANT AND DUAL SUBTRACTION GAMES RESOLVING THE DUCHÊNE-RIGO CONJECTURE.

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ABSTRACT. We prove a recent conjecture of Duchêne and Rigo, stating that every complementary pair of homogeneous Beatty sequences represents the solution to an *invariant* impartial game. Here invariance means that each available move in a game can be played anywhere inside the game-board. In fact, we establish such a result for a wider class of pairs of complementary sequences, and in the process generalize the notion of a *subtraction game*. Given a pair of complementary sequences (a_n) and (b_n) of positive integers, we define a game G by setting $\{\{a_n, b_n\}\}$ as invariant moves. We then introduce the invariant game $G^\star$, whose moves are all non-zero P-positions of G. Provided the set of non-zero P-positions of $G^\star$ equals $\{\{a_n, b_n\}\}$, this *is* the desired invariant game. We give sufficient conditions on the initial pair of sequences for this 'duality' to hold.

1. NOTATION, TERMINOLOGY AND STATEMENT OF RESULTS

This note concerns 2-person, impartial games (see [BCG]) played under normal (as against misère) rules. Let $\mathbb{N}$, $\mathbb{N}_0$ denote the positive and the non-negative integers respectively. For $k \in \mathbb{N}$, let $\mathcal{B} = \mathcal{B}(k) := (\mathbb{N}_0^k, \oplus, \preceq)$ denote the partially-ordered semigroup consisting of all ordered k-tuples of non-negative integers, where for elements $\boldsymbol{x} = (x_1, \ldots, x_k), \boldsymbol{y} = (y_1, \ldots, y_k)$ of $\mathcal{B}$ one defines

$$\boldsymbol{x} \oplus \boldsymbol{y} := (x_1 + y_1, \ldots, x_k + y_k)$$

and

$$\boldsymbol{x} \preceq \boldsymbol{y} \ \Leftrightarrow \ x_i \le y_i, \ i = 1, \ldots, k.$$

Hence $\boldsymbol{x} \prec \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $x_i < y_i$ for some i. For $\boldsymbol{y} \preceq \boldsymbol{x}$ we define

$$\boldsymbol{x} \ominus \boldsymbol{y} := (x_1 - y_1, \ldots, x_k - y_k).$$

We call $\mathcal{B}$ the *game board*. Let $G = G(F, \mathcal{B})$ denote a game, where for all $\boldsymbol{x} \in \mathcal{B}$, $F(\boldsymbol{x}) \subset \mathcal{B}$ defines the set of *options* of $\boldsymbol{x}$ in the sense that $\boldsymbol{y} \in F(\boldsymbol{x})$ if and only if there is a move from $\boldsymbol{x}$ to $\boldsymbol{y}$. Formally, the *move* from $\boldsymbol{x}$ to $\boldsymbol{y}$ is the ordered pair $(\boldsymbol{x}, \boldsymbol{y})$. In this paper, the phrase '$\boldsymbol{x} \to \boldsymbol{y}$ is an option' will often be used synonymously with '$\boldsymbol{y} \in F(\boldsymbol{x})$', in order to avoid cumbersome notation.

Given this setting, the two players only need to (randomly) pick a starting position $\boldsymbol{x} \in \mathcal{B}$ and decide who plays first. Then they play by alternating in choosing options from $F(\cdot)$ (and moving accordingly). Although we have announced that the last player to move wins (normal play), without some additional assumptions there is no guarantee that the game will terminate.

Key words and phrases. Beatty sequence, Complementary sequences, Impartial game, Invariant game, Superadditivity.

By a k*-pile subtraction game*[1] we mean a game played on $\mathcal{B}$ such that, for each $\boldsymbol{x} \in \mathcal{B}$, the set $F(\boldsymbol{x}) \subset \mathcal{B}$ has the property that $\boldsymbol{y} \in F(\boldsymbol{x}) \Rightarrow \boldsymbol{y} \prec \boldsymbol{x}$. In the setting of invariant games (to be defined below), it will be convenient to abuse notation and also refer to the k-tuple $\boldsymbol{x} \ominus \boldsymbol{y} \succ \boldsymbol{0}$ as a *move*. Observe that both options and moves are then elements of $\mathcal{B}$, but with different meanings.

In this paper, whenever we refer to a *(subtraction) game* we intend a k-pile subtraction game. Let G be a game. Then $\mathcal{T} = \mathcal{T}(G) := \{\boldsymbol{x} \mid F(\boldsymbol{x}) = \emptyset\}$ denotes the set of *terminal* positions. Clearly $\boldsymbol{0} := (0, \ldots, 0) \in \mathcal{T}$ and $\boldsymbol{0}$ is unique. It is natural to require that $\mathcal{T}$ be a *lower ideal* in the poset, that is, if $\boldsymbol{x} \in \mathcal{T}$ and $\boldsymbol{y} \prec \boldsymbol{x}$, then $\boldsymbol{y} \in \mathcal{T}$. Clearly, in this setting, any game must terminate within a finite number of moves and the *winner* is the player who makes the last move. The opponent is the *loser*.

Recently, Duchêne and Rigo [DR] introduced the notion of an invariant game.

A k-pile *invariant* subtraction game G is defined by a set $\mathcal{M}(G) \subseteq \mathcal{B} \setminus \{\boldsymbol{0}\}$ of (invariant) moves, where, for every $\boldsymbol{r} \in \mathcal{M}(G)$ and every $\boldsymbol{x} \succeq \boldsymbol{r},\ \boldsymbol{x} \to (\boldsymbol{x} \ominus \boldsymbol{r})$ is an option (and these are all the options)[2]. If a game is not invariant it is *variant*.

A position (a game) is P if all of its options are N. Otherwise it is N. This means that the first player to move wins if and only if the game is N. As usual, we shall denote by $\mathcal{P}(G)$ (resp. $\mathcal{N}(G)$) the collection of P- (resp. N-) positions of G.

Finally, if G is a (not necessarily invariant) game, then we can define an invariant game $G^\star$ on the same game board by setting

$$\mathcal{M}(G^\star) := \mathcal{P}(G) \backslash \{\boldsymbol{0}\}. \tag{1.1}$$

Example 1. Define G by $\mathcal{M}(G) = \emptyset$. Then $\mathcal{P}(G) = \mathcal{B}$ and so $\mathcal{M}(G^\star) = \mathcal{B} \setminus \{\boldsymbol{0}\}$. This gives $\mathcal{P}(G^\star) = \{\boldsymbol{0}\}$, so that in fact $\mathcal{N}(G^\star) = \mathcal{M}(G^\star)$. This latter equality does not hold in general. For example, let G rather denote 2-pile Nim. Then[3] $\mathcal{M}(G) = \{\{0, x\} \mid x \in \mathbb{N}\}$ and $\mathcal{P}(G) = \{\{x, x\} \mid x \in \mathbb{N}_0\}$. By (1.1), this gives $\mathcal{M}(G^\star) = \{\{x, x\} \mid x \in \mathbb{N}\}$. Then it is easy to see that $\mathcal{P}(G^\star) = \{\{0, x\} \mid x \in \mathbb{N}_0\}$. Hence, for the two games in this example we have that $(G^\star)^\star = G$. Neither does this equality hold in general. (See also Example 2.)

From now onwards we let $k = 2$.

A pair of sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ of positive integers is said to be *complementary* if $\{x_n\} \cup \{y_n\} = \mathbb{N}$ and $\{x_n\} \cap \{y_n\} = \emptyset$.

Let $\alpha < \beta$ be positive irrational numbers satisfying $1/\alpha + 1/\beta = 1$. Hence $1 < \alpha < 2 < \beta$. We call (α, β) an (ordered) *Beatty pair*. It is well-known [BOHA] that the sequences $(\lfloor n\alpha \rfloor)_{n \in \mathbb{N}}$ and $(\lfloor n\beta \rfloor)_{n \in \mathbb{N}}$ are complementary.

Our purpose is to prove the following conjecture [DR]:

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[1] Our subtraction games are generalizations of the Nim-type subtraction games defined in [BCG]. There are some alternative names for our games that can be found in the literature, such as *Take-away games, Removal games*. By our choice we emphasize the natural additive structure on $\mathcal{B}$.

[2] This notation and terminology is consistent with that employed in [DR].

[3] A subset R of $\mathcal{B} = \mathbb{N}_0 \times \mathbb{N}_0$ is *symmetric* if $(x, y) \in R \Leftrightarrow (y, x) \in R$. (We dispense with the obvious generalisation to $k > 2$ piles.) If the sets $\mathcal{M}(G)$ and $\mathcal{T}(G)$ are symmetric subsets of $\mathcal{B}$, then so are the sets $\mathcal{N}(G)$ and $\mathcal{P}(G)$. In this case the game G will be called *symmetric*. Sometimes it will be convenient to denote moves and positions of a symmetric game by unordered pairs $\{r, s\}$. Hence, whenever we write '$\{r, s\} \in \mathcal{M}(G)$' for example, what we mean is that $\{(r, s), (s, r)\} \subseteq \mathcal{M}(G)$.

Conjecture 1.1 (Duchêne-Rigo)**.** Let (α, β) be a Beatty pair. Then there exists an invariant game G such that $\mathcal{P}(G) = \{\{\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor\} \mid n \in \mathbb{N}_0\}$.

Let $t \in \mathbb{N}$. We say that a sequence $(X_n)_{n\in\mathbb{N}_0}$ of non-negative integers is t*-superadditive* if, for all $m, n \in \mathbb{N}_0$,

$$X_m + X_n \leq X_{m+n} < X_m + X_n + t. \tag{1.2}$$

Note that the left-hand inequality of (1.2) is the usual definition of *superadditivity*.

Let $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ be sequences of positive integers and define $a_0 = b_0 = 0$. We say that the set $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ of ordered pairs is t*-superadditive-complementary*, abbreviated t-SAC, if the following criteria are satisfied:

- $a_1 = 1$,
- a and b are complementary sequences,
- a is increasing,
- b is t-superadditive.

We can now state the main result of this paper :

Theorem 1.2. Suppose that the set $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ of ordered pairs is b_1-SAC. Define G by setting $\mathcal{M}(G) := \{\{a_n, b_n\} \mid n \in \mathbb{N}\}$. Then

$$\mathcal{P}(G^\star) = \mathcal{M}(G) \cup \{\mathbf{0}\} \tag{1.3}$$

and

$$(G^\star)^\star = G. \tag{1.4}$$

An immediate consequence of this result is

Corollary 1.3. Suppose that $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ is b_1-SAC. Then there is an invariant game I such that $\mathcal{P}(I) = \{\{a_n, b_n\} \mid n \in \mathbb{N}_0\}$.

Proof of Corollary. Take $I = G^\star$ in Theorem 1.2. □.

It is well-known and easy to check that if a and b are a pair of complementary homogeneous Beatty sequences, then the set $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ is 2-SAC, hence b_1-SAC. Therefore, Corollary 1.3 implies Conjecture 1.1.

Because of (1.4), it is natural to refer to the game $G^\star$ defined by (1.1) as the *dual* of G, when G satisfies the hypotheses of Theorem 1.2. It is important to note, however, that the 'duality relation' (1.4) doesn't always hold for games G not satisfying these hypotheses.

Example 2. As a simple but instructive example, take $G = \text{WN}$, the ordinary Wythoff Nim game [W], so that $\mathcal{M}(\text{WN}) = \{\{0, i\}, (i, i) \mid i \in \mathbb{N}\}$. This set obviously does not satisfy the hypotheses of Theorem 1.2, whereas $\mathcal{M}(\text{WN}^\star)$ does so. Indeed, according to (1.1), we have

$$\mathcal{M}(\text{WN}^\star) = \mathcal{P}(\text{WN})\backslash\{\mathbf{0}\} = \{\{\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor\} \mid n \in \mathbb{N}\}, \quad \phi = \frac{1+\sqrt{5}}{2}. \tag{1.5}$$

It is easy to see that $\{\{0, x\} \mid x \in \mathbb{N}_0\} \subset \mathcal{P}(\text{WN}^\star)$. Otherwise it is easy to check that the P-positions of $\text{WN}^\star$ begin

$$(1, 1), (3, 3), (3, 4), (4, 4), (6, 6), (8, 8), (8, 9), (8, 12), (9, 9), (9, 12), \ldots$$

(see Figure 1 and [L3] for further results) and hence

$$(\mathrm{WN}^\star)^\star \neq \mathrm{WN}.$$

But if we go one step further, it follows from (1.5) and Theorem 1.2 that

$$((\mathrm{WN}^\star)^\star)^\star = \mathrm{WN}^\star.$$

In particular, the games WN and $(\mathrm{WN}^\star)^\star$ do have the same P-positions.

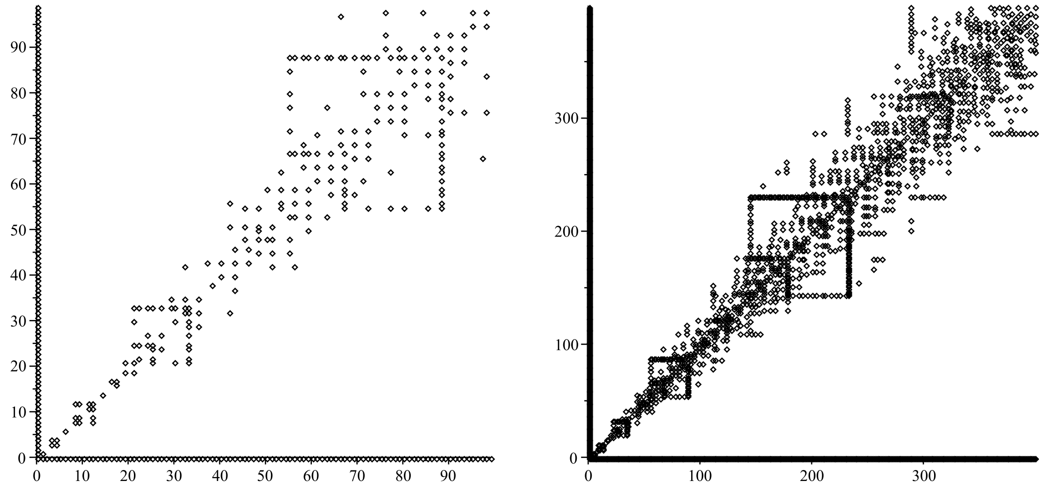

FIGURE 1. The set $\{\{i,j\} \in \mathcal{P}(\mathrm{WN}^\star) \mid 0 \le i,j \le x\} = \{(0,0)\} \cup \{\{i,j\} \in \mathcal{M}((\mathrm{WN}^\star)^\star) \mid 0 \le i,j \le x\}$, for $x = 100, 400$ respectively.

Numerous generalizations and variations of Wythoff Nim can be found in the literature. In fact, this game can be credited with opening up the territory of the games we are exploring in this paper. However, we have not been able to find any literature on the game $(\mathrm{WN}^\star)^\star$.

The rest of the paper is organised as follows. In Section 2, we will prove Theorem 1.2. In Section 3, we explore the problem of describing necessary and sufficient conditions on a pair $(a_n), (b_n)$ of complementary sequences for there to exist an invariant game G with $\mathcal{P}(G) = \{\{a_n, b_n\}\} \cup \{\mathbf{0}\}$. We are unable to solve this problem definitively, though we discuss several pertinent examples. One of these concerns an application of Theorem 1.2 to defining an invariant game with the same solution as the variant game 'the Mouse game' [F3]. In another example, we study the set of P-positions of the invariant game $G = (1,2)$GDWN [L2]. Here, the b-sequence is not increasing and we show that $\mathcal{P}((G^\star)^\star) \neq \mathcal{P}(G)$.

2. PROOF OF THEOREM 1.2

Let us begin by proving some basic facts about any sequence of b_1-SAC pairs.

Proposition 2.1. Suppose that $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ is b_1-SAC. Then, for all $n \in \mathbb{N}_0$,

(i) $b_{n+1} - b_n \ge b_1 \ge 2$,

(ii) $a_{n+1} - a_n \in \{1, 2\}$,
(iii) $a_n < b_n$ and the sequence $(b_n - a_n)$ is non-decreasing,
(iv) for all $m, n \in \mathbb{N}_0$,

$$a_m + a_n - 1 \leq a_{m+n} \leq a_m + a_n + 1. \tag{2.1}$$

Proof. Part (i): By definition $a_1 = 1$. Then, by complementarity, $b_1 \geq 2$. The first inequality follows by superadditivity.
Part (ii): Let $d_n := a_{n+1} - a_n$. Since a is increasing we have $d_n \geq 1$ for all n. Suppose that there exists an n such that $d_n \geq 3$. Then, by complementarity, there exists an i such that $b_i = a_n + 1$ and $b_{i+1} = a_n + 2$. But then $b_{i+1} - b_i = 1$, contradicting (i).
Part (iii): We have $b_1 > a_1$ by definition, and it follows from parts (i) and (ii) that the sequence $(b_n - a_n)$ is non-decreasing.
Part (iv): Note that, since the sequences (a_i) and (b_i) are increasing and complementary, we have for any $i > 0$ that

$$b_{a_i - i} < a_i < b_{a_i - i + 1}. \tag{2.2}$$

The inequalities in (2.1) are trivial if either m or n equals zero, so we may suppose that $m, n > 0$. Fix m and n. Let the integers r, s be defined by

$$b_r < a_m < b_{r+1}, \qquad b_s < a_n < b_{s+1}. \tag{2.3}$$

Then, by (2.2), it follows that $a_m = m + r$ and $a_n = n + s$, hence that $a_m + a_n = (m + n) + (r + s)$. First of all, consider the right-hand inequalities in (2.3). Superadditivity of b implies that

$$b_{r+s+2} \geq b_{r+1} + b_{s+1} \geq a_m + a_n + 2 = (m + n) + (r + s + 2).$$

Then, by (2.2) again we must have

$$a_{m+n} \leq (m + n) + (r + s + 1) = a_m + a_n + 1,$$

which proves the right-hand inequality of (2.1).

Secondly, the fact that the sequence b is b_1-superadditive implies that

$$b_{r+s-1} \leq b_{r-1} + b_s + (b_1 - 1) \leq (b_r - b_1) + b_s + (b_1 - 1) = b_r + b_s - 1.$$

This, together with the left-hand inequalities in (2.3), imply that

$$b_{r+s-1} \leq (a_m - 1) + (a_n - 1) - 1 = (m + n) + (r + s - 3).$$

By complementarity, it follows that

$$a_{m+n-2} \geq (m + n) + (r + s - 3).$$

Then, the fact that a is increasing implies that

$$a_{m+n} \geq a_{m+n-2} + 2 \geq (m + n) + (r + s - 1) = a_m + a_n - 1,$$

which proves the left-hand inequality of (2.1). This completes the proof of Proposition 2.1. □

Remark 1. In the above proof, superadditivity of b sufficed, except for the left-hand inequality in (2.1). Only the latter required b_1-superadditivity. Interestingly enough, b_1-superadditivity is needed for the proof of Theorem 1.2, but the left-hand inequality in (2.1) is not.

For our particular setting, the next lemma is a special case of part (iii) of the one to follow. But it is nice to first state it in a more general form.

Lemma 2.2 (A P-position is never an invariant move)**.** Let G be an invariant subtraction game. Then $\mathcal{M}(G) \cap \mathcal{P}(G) = \emptyset$.

Proof. Suppose that there was a move $\boldsymbol{r} \in \mathcal{P}(G)$. Then, in particular, $\mathbf{0} = \boldsymbol{r} - \boldsymbol{r} \in F(\boldsymbol{r})$. But $\mathbf{0} \in \mathcal{P}(G)$, so then $\boldsymbol{r} \in \mathcal{N}(G)$, a contradiction. □

The hypothesis of the next lemma is satisfied, in particular, by any game G for which $\mathcal{M}(G) \cup \{(\mathbf{0})\}$, viewed as an ordered set, is b_1-SAC. The items (i) and (ii) characterize precisely the lower ideal $\mathcal{T}(G)$.

Lemma 2.3. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be any pair of increasing sequences of positive integers, and suppose that G is an invariant subtraction game with $\mathcal{M}(G) = \{\{a_n, b_n\}\}$. Then

- (i) $\{0, k\} \in \mathcal{P}(G)$, for all $k \in \mathbb{N}_0$,
- (ii) $\{k, l\} \in \mathcal{P}(G)$ if $k, l \in \{1, 2, \ldots, b_1 - 1\}$,
- (iii) If $k, l > 0$ then $\{k, l\} \in \mathcal{N}(G)$ if, for some $n > 0$,
  - (a) $k = a_n$ and $l \geq b_n$, or
  - (b) $k = b_n$ and $l \geq a_n$, or
  - (c) $a_{n-1} \leq k < a_{n-1} + b_1$ and $b_{n-1} \leq l < b_{n-1} + b_1$.

Proof. Parts (i), (ii): By the definition of $\mathcal{M}(G)$, it is clear that $F(\{k, l\}) = \emptyset$ if either $\min\{k, l\} = 0$ or $\max\{k, l\} < b_1$.
Part (iii): If (a) holds, then

$$(k, l) \to (k, l) \ominus (a_n, b_n) = (0, l - b_n),$$

is an option in G. Since $(0, l - b_n) \in \mathcal{P}(G)$ by (i), it follows that $(k, l) \in \mathcal{N}(G)$. Similarly, if (b) holds then one considers the option

$$(k, l) \to (k, l) \ominus (b_n, a_n) = (0, l - a_n) \in \mathcal{P}(G).$$

Finally, if (c) holds, then we have the option

$$(k, l) \to (k, l) \ominus (a_{n-1}, b_{n-1}) = (k - a_{n-1}, l - b_{n-1}).$$

Since $k - a_{n-1} < b_1$ and $l - b_{n-1} < b_1$, we have $(k - a_{n-1}, l - b_{n-1}) \in \mathcal{P}(G)$ by (ii), and hence $(k, l) \in \mathcal{N}(G)$ once more. □

Proof of Theorem 1.2. Clearly, (1.4) follows from (1.3) so it remains to prove the latter. Recall that the moves in the game $G^\star$ are given by $\mathcal{M}(G^\star) := \mathcal{P}(G) \setminus \{\mathbf{0}\}$ and where $\mathcal{M}(G) := \{\{a_n, b_n\} \mid n \in \mathbb{N}_0\} \setminus \{\mathbf{0}\}$. We want to show that

$$\mathcal{P}(G^\star) = \{\{a_n, b_n\} \mid n \in \mathbb{N}_0\}. \tag{2.4}$$

By the definition of $\mathcal{P}$, this corresponds to showing that, for all $(\alpha, \beta) \in \mathcal{B}$,

$$\exists\, n \text{ such that either } (\alpha, \beta) \to (a_n, b_n) \text{ or } (\alpha, \beta) \to (b_n, a_n) \text{ is an option in } G^\star \tag{2.5}$$

if and only if $\{\alpha, \beta\} \neq \{a_i, b_i\}$ for all $i \in \mathbb{N}_0$.

"N→ P": Suppose that $\{\alpha, \beta\} \neq \{a_i, b_i\}$ for any $i \in \mathbb{N}_0$. If $(\alpha, \beta) \in \mathcal{P}(G)$ then $(\alpha, \beta) \to \mathbf{0} = (a_0, b_0)$ is an option in $G^\star$, thus satisfying (2.5). If $(\alpha, \beta) \in \mathcal{N}(G)$, then there exists $(x, y) \in \mathcal{P}(G)$ such that $(\alpha, \beta) \to (x, y)$ is an option in G. By definition of $\mathcal{M}(G)$, there exists $j \in \mathbb{N}$ such that either $(\alpha, \beta) \ominus (a_j, b_j) = (x, y)$ or $(\alpha, \beta) \ominus (b_j, a_j) = (x, y)$. Note that our assumptions thus imply that $(x, y) \neq \mathbf{0}$. Hence $(x, y) \in \mathcal{P}(G) \backslash \{\mathbf{0}\} = \mathcal{M}(G^\star)$. Since $(\alpha, \beta) \ominus (x, y) \in \{(a_j, b_j), (b_j, a_j)\}$, we see that once again (2.5) is satisfied.

"P$\to$ N": Suppose that $\{\alpha, \beta\} = \{a_i, b_i\}$ for some $i \in \mathbb{N}_0$ and that (2.5) holds. By symmetry, it suffices to consider the following two cases : there exists $m, n \in \mathbb{N}_0$ such that $m > 0$ and either $(a_{m+n}, b_{m+n}) \to (a_n, b_n)$ or $(a_{m+n}, b_{m+n}) \to (b_n, a_n)$ is an option in $G^\star$.

First suppose the latter. Let

$$(x, y) := (a_{m+n}, b_{m+n}) \ominus (b_n, a_n).$$

By definition of $G^\star$, we must have $(x, y) \in \mathcal{P}(G) \backslash \{\mathbf{0}\}$. By Lemma 2.2, we may assume that $n > 0$. Then $x = a_{m+n} - b_n < a_{m+n} - a_n \leq a_m + 1$, by parts (iii) and (iv) of Proposition 2.1. Hence $x \leq a_m$. By complementarity, there exists $p \leq m$ such that $x \in \{a_p, b_p\}$. On the other hand, $y = b_{m+n} - a_n > b_{m+n} - b_n \geq b_m$, by superadditivity of b. In particular, $y > x$. But then $(x, y) \in \mathcal{N}(G)$, by parts (a),(b) of Lemma 2.3(iii), a contradiction.

Second, suppose that $(a_{m+n}, b_{m+n}) \to (a_n, b_n)$ is an option in $G^\star$. Let

$$(x, y) := (a_{m+n}, b_{m+n}) \ominus (a_n, b_n). \tag{2.6}$$

As before, we must prove the contradiction that $(x, y) \in \mathcal{N}(G)$. By the b_1-superadditivity of b, we have

$$b_m \leq y < b_m + b_1. \tag{2.7}$$

If $x \leq a_m$ then we can appeal to parts (a),(b) of Lemma 2.3(iii) again. By the right-hand inequality of (2.1), the only other possibility is that $x = a_m + 1$. Since $m > 0$ and $y \geq b_m$, we have $y \geq x$. If $x = b_i$ for some i, then part (b) of Lemma 2.3(iii) gives a contradiction. This leaves the possibility that $x = a_{m+1} = a_m + 1$. But then, because of (2.7), we get a contradiction from part (c) of Lemma 2.3(iii). □

3. DISCUSSION

In this section we provide four examples and suggest some future work.

Example 3. Let a and b be any complementary, though not necessarily increasing, sequences beginning as in Table 1 below.

As usual, set $a_0 = b_0 := 0$. Note that the set of pairs $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ cannot be b_1-SAC, since $b_3 = b_{2+1} = b_2 + b_1 + b_1$. Suppose there were an invariant game G with $\mathcal{P}(G) = \{\{a_n, b_n\} \mid n \in \mathbb{N}_0\}$. Then $(2, 6) \in \mathcal{N}(G)$. But $(2, 6) = (4, 13) \ominus (2, 7)$, a contradiction.

Nevertheless, if the sequences a and b are increasing, $a_1 = 1$ and the b-sequence grows at only a slightly faster rate than that allowed by (1.2), then Theorem 1.2 will hold again. Indeed, suppose that

$$b_2 \geq 2b_1 \text{ and } b_{m+n} \geq b_{m+1} + b_n \text{ for all } m \geq 1, n \geq 2. \tag{3.1}$$

We can still use Lemma 2.3 and one may check that the proof of Theorem 1.2 goes through. Consider (2.6), for example. We still have $x \leq a_m + 1 \leq a_{m+1}$, since for the

b_n	3	7	13
a_n	1	2	4
n	1	2	3

TABLE 1. The b-sequence does not satisfy the right-hand inequality in (1.2).

right-hand inequality of (2.1) we only required b to be superadditive. If $n \geq 2$, then (3.1) implies that $y \geq b_{m+1}$. Then from Lemma 2.3(iii), parts (a) and (b), it follows that $(x, y) \in \mathcal{N}(G)$. We can obtain the same conclusion even when $n = 1$, since then we still have $y \geq b_m$ and now $x = a_{m+1} - 1 < a_{m+1}$, with strict inequality.

Example 4. (A similar example to this one appears in [DR]). Let a and b be any complementary sequences beginning as in Table 2.

Put $a_0 = b_0 := 0$. The set of pairs $\{(a_n, b_n) \mid n \in \mathbb{N}_0\}$ cannot be b_1-SAC since $b_2 = b_{1+1} = b_1 + b_1 - 1$. Suppose there were an invariant game G with $\mathcal{P}(G) = \{\{a_n, b_n\} \mid n \in \mathbb{N}_0\}$. Then $(1, 3) \in \mathcal{N}(G)$. But $(1, 3) = (2, 7) \ominus (1, 4)$, a contradiction.

b_n	4	7
a_n	1	2
n	1	2

TABLE 2. The b-sequence does not satisfy superadditivity, the left-hand inequality in (1.2).

This example also arises from a pair of complementary, but inhomogeneous Beatty sequences. Let (α, β) be a Beatty pair. Let $\gamma, \delta \in \mathbb{R}$. For each $n \in \mathbb{N}$, let

$$a_n := \lfloor n\alpha + \gamma \rfloor, \qquad b_n := \lfloor n\beta + \delta \rfloor. \tag{3.2}$$

Fraenkel [F1] proved that the sequences (a_n) and (b_n) are complementary if and only if $n\beta + \delta \notin \mathbb{Z}$ for any $n \geq 1$, and

$$\frac{\gamma}{\alpha} + \frac{\delta}{\beta} = 0. \tag{3.3}$$

Choose a pair of (small) irrational numbers $\epsilon_1, \epsilon_2 > 0$. Let $\alpha := \frac{7}{5} + \epsilon_1$, $\beta := \frac{7}{2} - \epsilon_2$. Choose $\delta \notin \mathbb{Q}(\beta)$ satisfying

$$\frac{1}{2} + \epsilon_2 \leq \delta < 1 - 2\epsilon_2. \tag{3.4}$$

It is not hard to check that, for an appropriate choice of $\epsilon_1, \epsilon_2, \delta$, the number $\gamma < 0$ defined by (3.3) will satisfy

$$-\frac{2}{5} - \epsilon_1 \leq \gamma < \frac{1}{5} - 2\epsilon_1. \tag{3.5}$$

From (3.4) and (3.5), one may then verify in turn that the sequences (a_n) and (b_n) defined by (3.2) begin as in Table 2.

Example 5. For each $n \in \mathbb{N}$, let $a_n := \lfloor \frac{3n}{2} \rfloor$ and $b_n := 3n - 1$. It is easy to see that (a_n) and (b_n) are a pair of complementary, inhomogeneous Beatty sequences. Put $a_0 = b_0 := 0$, as usual. In [F3], a variant game G named 'the Mouse game' was invented with $\mathcal{P}(G) = \{\{a_n, b_n\} \mid n \in \mathbb{N}_0\}$. But, since it is easy to verify that $\{(a_n, b_n)\}$ is b_1-SAC, by Theorem 1.2 we may also introduce an invariant game H, which we call 'the Mouse trap', with $\mathcal{P}(H) = \mathcal{P}(G)$. In analogy with Example 2, the invariant rules are $\mathcal{M}(H) = \mathcal{P}(G^\star)$.

Remark 2. In [F2, L1] invariant games with symmetric moves are defined whose P-positions consist of complementary inhomogeneous Beatty sequences (CIBS). Both papers include variations of Wythoff Nim. In the former a misère variation (the player who moves last loses) is studied. Indeed, we believe it to be the 'most natural/direct' way to construct a game with CIBS as P-positions. In the latter paper, the terminal positions are $(l, 0)$ and $(0, p - l)$, for some integers $0 < l < p$, so the game is only symmetric if $p = 2l$. Namely, here the game board is rearranged to

$$\mathcal{B} := (\mathbb{N}_0 \times \mathbb{N}_0) \backslash \{(i, j) \mid 0 \leq i < l, 0 \leq j < p - l\}.$$

The above examples provide some extra insight into the following problem, which nevertheless remains wide open :

Problem 1. Let (a_n), (b_n) be a pair of complementary, increasing sequences with $a_1 = 1$. Find necessary and sufficient conditions for the existence of an invariant game G with $\mathcal{P}(G) = \{\{a_n, b_n\}\} \cup \{\mathbf{0}\}$.

A special case which might be more tractable is the case of inhomogeneous Beatty sequences. Motivated by Examples 4 and 5, we may ask

Problem 2. Let (a_n), (b_n) be a pair of complementary, inhomogeneous Beatty sequences with $a_1 = 1$. Is it true that there exists an invariant game G with $\mathcal{P}(G) = \{\{a_n, b_n\}\} \cup \{\mathbf{0}\}$ if and only if the set of pairs $\{(a_n, b_n)\}$ is b_1-SAC ?

Combining Theorem 1.2 with Example 2 (Wythoff Nim) leads us to the following question.

Problem 3. Let (a_n) and (b_n) be a pair of complementary, increasing sequences with $a_1 = 1$. Suppose further that there exists an invariant subtraction game G with $P(G) = \{\{a_n, b_n\}\} \cup \{0\}$. Is then $P((G^\star)^\star) = P(G)$?

We know that the answer to Problem 3 is no, if we drop the condition that (b_n) is increasing. Consider the following example :

Example 6. Let G be the invariant game $(1, 2)$GDWN, studied in [L2], so that $\mathcal{M}(G) = \{\{0, i\}, (i, i), \{i, 2i\} \mid i \in \mathbb{N}\}$. Define

$$\{\{a_n, b_n\} \mid n \in \mathbb{N}_0\} := \mathcal{P}(G), \quad \text{where } (a_n) \text{ is increasing.}$$

Then the sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are complementary, but b is not increasing. Table 3 gives the initial P-positions of this game.

b_n	0	3	6	5	10	14	17	25	28	18	35	23
a_n	0	1	2	4	7	8	9	11	12	13	15	16
n	0	1	2	3	4	5	6	7	8	9	10	11

TABLE 3. The initial P-positions of the game $(1,2)$GDWN. For a more comprehensive list, see [L2].

Now consider the game $G^\star$, as defined by (1.1). It is not hard to check that $(11,23) \in \mathcal{P}(G^\star)$. However, by brute-force calculation one may also verify that $(104,235)$ and $(115,258)$ are in $\mathcal{P}(G)$. Since

$$(115,258) \ominus (104,235) = (11,23),$$

we see that $\mathcal{P}((G^\star)^\star)$ cannot coincide with $\mathcal{P}(G)$.

Another possible direction for future work is to extend our results in some manner to k-pile subtraction games for $k > 2$, or even perhaps to consider subtraction games played on other partially-ordered semigroups. Alternatively, one might try to extend the notion of 'invariance' to games which cannot be formulated as subtraction games. Many such games appear in the literature, see for example [S], where 14 such games are proved Pspace-complete, 3 played on graphs, including Geography — whose many variations have been addressed in other papers — and 11 on propositional formulas. Another example is annihilation games — if a token moves onto another one, both disappear — for which there is a polynomial-time winning strategy [FY].

Finally, the "$\star$-operator" introduced in (1.1) and the duality in (1.4) may turn out to be useful in other contexts.

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