From heaps of matches to the limits of computability

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Chalmers University of Technology,
CANT 2011

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From heaps of matches to the limits of computability
Place a heap of tokens on a table

**Figure:** Two player’s alternate in removing tokens. Last player wins.
The move options

**Figure:** Rules: remove “one”, “two” or “three” tokens.
How to win

**Figure:** The Previous player’s safe positions are divisible by four.
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A winning move

**Figure:** P-positions divisible by four.
The new move options are losing

**Figure:** P-positions divisible by four.
Periodic P-positions

- If the set of numbers allowed to remove from the heap is finite,
Periodic P-positions

- If the set of numbers allowed to remove from the heap is finite,
- the set of P-positions will ultimately become periodic.
Periodic P-positions

- If the set of numbers allowed to remove from the heap is finite,
- the set of P-positions will ultimately become periodic.
- The game cannot embrace any mysteries.
Several heaps

- A fixed number $d$ of heaps,
Several heaps

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- a position can be regarded as a $d$-dimensional vector $x = (x_1, \ldots, x_d)$ of non-negative integers.
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- The rules of the game are encoded by a finite set $\mathcal{M}$ of integer vectors that specify the permitted moves.
Several heaps

- A fixed number $d$ of heaps,
- a position can be regarded as a $d$-dimensional vector $x = (x_1, \ldots, x_d)$ of non-negative integers.
- The rules of the game are encoded by a finite set $\mathcal{M}$ of integer vectors that specify the permitted moves.
- If $(m_1, \ldots, m_d) \in \mathcal{M}$, then from position $(x_1, \ldots, x_d)$, a player can move to position $(x_1 + m_1, \ldots, x_d + m_d)$, provided all of the numbers $x_1 + m_1, \ldots, x_d + m_d$ are non-negative.
Invariant games and termination

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- The move options given by $M$ are independent of the position played from,
Invariant games and termination

- **Invariant** games,
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- If some moves involve adding tokens to heaps, the game does not necessarily have to terminate.
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- In each move, the total number of tokens decreases.
Can we understand the P-positions?

- Each game $M$ has a set $P(M)$ of P-positions,
Can we understand the P-positions?

- Each game $\mathcal{M}$ has a set $\mathcal{P}(\mathcal{M})$ of P-positions,
- From the list $\mathcal{M}$ of permitted moves, is it possible to completely understand $\mathcal{P}(\mathcal{M})$?
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- For any given position $x = (x_1, \ldots, x_d)$ we can determine recursively whether or not $x$ is a P-position by first computing the status of all positions with fewer tokens.
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- Yes, at least in one sense:
- For any given position $x = (x_1, \ldots, x_d)$ we can determine recursively whether or not $x$ is a P-position by first computing the status of all positions with fewer tokens.
- The position $x$ is in $P(M)$ iff there is no move from $x$ to a position in $P(M)$.
Patterns and P-positions

Figure: The P-positions of the game $\mathcal{M} = \{(-1, -3), (-2, 1)\}$. 
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Figure: A pattern that occurs in $\mathcal{P}$ an one that does not
Patterns and P-positions

Figure: Pattern occurrence
Patterns and P-positions

Figure: Pattern occurrence.
Undecidable games

- In general, the P-positions of a given game are much harder to understand.
Undecidable games

- In general, the P-positions of a given game are much harder to understand.
- If we cannot answer questions of pattern occurrence, we cannot claim to have full understanding of the set of P-positions of a game.
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Games of more than one heap
Cellular automata
An undecidable class of games of two heaps
Invariance
Undecidability of game equivalence?

\[ M = \{(0, -2), (-2, 0), (2, -3), (-3, 2), (-3, -3)\} \]
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\[ M = \{(0, -2), (-2, 0), (2, -3), (-3, 2), (-5, 4), (-5, -2), (-4, -3), (-1, -4)\} \]
Undecidable games

From the list $\mathcal{M}$ of permitted moves, it is not possible to completely understand $P(\mathcal{M})$ in the following sense:
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**Theorem**

*There is no algorithm that, given as input the set of moves $\mathcal{M}$ and a finite pattern, decides whether the given pattern occurs in $P(\mathcal{M})$.***
Cellular automata

- Cellular automata give rise to similar 2-dimensional patterns.

It is known that some questions including pattern-occurrence are algorithmically undecidable. A restricted class of CA's for which the undecidability results are known to still hold. Two states (0 and 1), and the state of cell \( i \) at time \( t \) is denoted by \( x^t_i \). The starting configuration is...

000111...

( the first `1' at position 0).
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- The update rule, a number $n$ and a Boolean function $f$ taking $n$ bits of input.
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- The states are updated by

$$x_{t+1,i} = f(x_{t,i-n+1}, x_{t,i-n+2}, \ldots, x_{t,i}),$$
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- $x_{t+1,i}$ depends on $x_{t,i}$ and the $n - 1$ cells immediately to the left of $x_{t,i}$. 

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- The update rule, a number \( n \) and a Boolean function \( f \) taking \( n \) bits of input.
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x_{t+1,i} = f(x_{t,i-n+1}, x_{t,i-n+2}, \ldots, x_{t,i}),
\]

- \( x_{t+1,i} \) depends on \( x_{t,i} \) and the \( n - 1 \) cells immediately to the left of \( x_{t,i} \).
- We require that \( f(0, \ldots, 0) = 0 \), so that \( x_{t,i} = 0 \) whenever \( i < 0 \).
Example

- $n = 2$ and letting the Boolean function be

$$f(x, y) = x \oplus y,$$
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- The 1’s are represented by red squares.
Example

- \( n = 2 \) and letting the Boolean function be
  
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- \( f(x, y) \) is equal to 0 if \( x = y \) and 1 if \( x \neq y \).
- The 1’s are represented by red squares.
- The leftmost column represents \( x_t,0 \).
Example

Figure: The CA given by \( f(x, y) = x \oplus y \) (Wolfram’s rule 90).
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The pattern is Pascal’s triangle modulo 2

**Figure**: The CA given by $f(x, y) = x \oplus y$ (Wolfram’s rule 90).
A well-known undecidability result

Pascal’s triangle modulo 2 is non-periodic, but well understood. There are Boolean functions that give rise to more difficult behavior (e.g. Wolfram’s Rule 110). The following theorem is well-known.
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**Theorem**

There is no algorithm that takes input $n$, $f$, and a bit-string $s$ of length $n$, and answers whether or not $s$ ever occurs in the CA given by $f$. 
CA: Rule 110

The CA given by Stephen Wolfram’s rule 110 and a (doubly) periodic initial pattern was proved undecidable by Matthew Cook around year 2000.
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- Update rule: “0s are changed to 1s at all positions where the value to the right is a 1, while 1s are changed to 0s at all positions where the values to the left and right are both 1”.
CA, rule 110, time moves upwards
Outline: The CA’s equivalence with games

- Define a class of non-invariant games and show that arbitrary CA’s can be reduced to computing their P-positions.
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- The tape-heap: its number of tokens corresponds to a position on the tape
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  - The tape-heap: its number of tokens corresponds to a position on the tape
  - The Time heap: we insert some space, $k$, to allow for the rules of the game to “compute” an arbitrary Boolean function $f$. 
Outline: The CA’s equivalence with games

- Define a class of non-invariant games and show that arbitrary CA’s can be reduced to computing their P-positions.
- Two heaps, the tape-heap and the time-heap:
  - The tape-heap: its number of tokens corresponds to a position on the tape
  - The Time heap: we insert some space, $k$, to allow for the rules of the game to “compute” an arbitrary Boolean function $f$.
- The positions with $km$ tokens in the time-heap will correspond to the state of the CA at time $m$. 
The permitted moves depend on the congruence class of the time heap

- Definition of a class of modular games.
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- Definition of a class of modular games.
- For some positive integer $k$, depending on our move function $f$, there are sets of integer vectors $\mathcal{M}_0, \ldots, \mathcal{M}_{k-1}$ that specify the available move options.
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- Definition of a class of modular games.
- For some positive integer $k$, depending on our move function $f$, there are sets of integer vectors $M_0, \ldots, M_{k-1}$ that specify the available move options.
- Let $(a_1, a_2)$ denote a given position, where $a_1$ is the number of tokens in the tape-heap and $a_2$ is the number of tokens in the time-heap,
The permitted moves depend on the congruence class of the time heap

- Definition of a class of modular games.
- For some positive integer $k$, depending on our move function $f$, there are sets of integer vectors $M_0, \ldots, M_{k-1}$ that specify the available move options.
- Let $(a_1, a_2)$ denote a given position, where $a_1$ is the number of tokens in the tape-heap and $a_2$ is the number of tokens in the time-heap,
- Then the set of available moves is given by $M_i$, where $i \equiv a_2 \pmod{k}$.

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The construction of a modular game computer

- Let square brackets \([\cdot]\) denote \(1 - \max(\cdot)\).
The construction of a modular game computer

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- For instance
  \[
  [xyz] = 1 - \max(x, y, z),
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  and by convention \([\ ] = 1\).
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- The set consisting of $\&$ and $\neg$ (negation) is a complete set of connectives in propositional logic.
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- Let square brackets [·] denote $1 - \max(·)$.
- For instance
  \[ [xyz] = 1 - \max(x, y, z), \]
  and by convention $[] = 1$.
- Every Boolean function can be expressed in terms of nested brackets.
- The set consisting of $\&$ and $\sim$ (negation) is a complete set of connectives in propositional logic.
- For instance the function $f(x, y) = x \oplus y$ can be expressed as
  \[ x \oplus y = [[xy][[x][y]]]. \]
Computing the function $f$

$$\left[\left[xy\right]\left[\left[x\right]\left[y\right]\right]\right]$$

**Figure:** A modular game computing $f(x, y) = x \oplus y$ in five steps. The arrows indicate move options. The value of each cell is the bracket of the values of all its options.
Computing the function $f$

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and its moves

The construction in the last figure corresponds to the modular game

\[ M_1 = \{ (-1, -1) \} \]
and its moves

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- $\mathcal{M}_2 = \{(0, -2)\}$
- $\mathcal{M}_1 = \{(-1, -1)\}$
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- $M_3 = \{(0, -3), (-1, -3)\}$
- $M_2 = \{(0, -2)\}$
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- $M_4 = \{(0, -2), (0, -3)\}$
- $M_3 = \{(0, -3), (-1, -3)\}$
- $M_2 = \{(0, -2)\}$
- $M_1 = \{(-1, -1)\}$
The construction in the last figure corresponds to the modular game

- $\mathcal{M}_0 = \{(0, -1), (0, -2)\}$
- $\mathcal{M}_4 = \{(0, -2), (0, -3)\}$
- $\mathcal{M}_3 = \{(0, -3), (-1, -3)\}$
- $\mathcal{M}_2 = \{(0, -2)\}$
- $\mathcal{M}_1 = \{(-1, -1)\}$
The P-positions

Figure: A modular game emulating $f(x, y) = x \oplus y$. Here the P-positions with fewer than 50 tokens in each heap are represented by filled squares. Rows corresponding to $a_2 \equiv 0 \pmod{5}$ are highlighted by drawing the P-positions in red.
A gadget keeps track of the congruence class of the time-heap

- Our invariant game has a time-heap, a tape-heap and a gadget with \( k \) more heaps.
A gadget keeps track of the congruence class of the time-heap

- Our invariant game has a time-heap, a tape-heap and a gadget with $k$ more heaps.
- Assume that one of the heaps in the gadget contains a single match and the remaining $k - 1$ heaps are empty.
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- Our invariant game has a time-heap, a tape-heap and a gadget with \( k \) more heaps.
- Assume that one of the heaps in the gadget contains a single match and the remaining \( k - 1 \) heaps are empty.
- This need not be the case, but we want a single match in the gadget to follow the time-heap modulo \( k \).
A gadget keeps track of the congruence class of the time-heap

- Our invariant game has a time-heap, a tape-heap and a gadget with $k$ more heaps.
- Assume that one of the heaps in the gadget contains a single match and the remaining $k - 1$ heaps are empty.
- This need not be the case, but we want a single match in the gadget to follow the time-heap modulo $k$.
- The heaps of the gadget may be numbered $0, \ldots, k - 1$. 
The gadget allows us to emulate a modular game by an invariant game

- For each of the move sets $M_i$ of the modular game, we introduce corresponding moves in the invariant game:
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- For each of the move sets $M_i$ of the modular game, we introduce corresponding moves in the invariant game:
  - The $i$th heap of the gadget is emptied,
  - The tape- and time-heaps are affected as given by $M_i$,
  - A match is added to the heap of the gadget corresponding to the new modulus of the time-heap.
Triviality ruled out

- In this way, the single token in the gadget follows the congruence class of the time heap.
Triviality ruled out

- In this way, the single token in the gadget follows the congruence class of the time heap.
- A subset of positions of the invariant game emulates the modular game, namely those where
  
(i) there is exactly one match in the gadget, and
(ii) this match is in the heap corresponding to the number of tokens in the time-heap modulo \(k\).

We may conclude that the pattern-occurrence problem is computationally unsolvable for the invariant game if and only if the patterns we ask for do not trivially occur for positions violating (i) or (ii).

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(i) there is exactly one match in the gadget, and

(ii) this match is in the heap corresponding to the number of tokens in the time-heap modulo $k$.

We may conclude that the pattern-occurrence problem is computationally unsolvable for the invariant game

if and only if the patterns we ask for do not “trivially” occur for positions violating (i) or (ii).
If the gadget contains more than one match

- All earlier moves will still be available,
If the gadget contains more than one match

- All earlier moves will still be available,
- Choose some large number $N$, and allow any move that transfers two tokens in the gadget to two other heaps and removes any number smaller than $N$ from the two main heaps.
If the gadget contains more than one match

- All earlier moves will still be available,
- Choose some large number $N$, and allow any move that transfers two tokens in the gadget to two other heaps and removes any number smaller than $N$ from the two main heaps.
- Since the number of tokens in the gadget will never change, this will give a trivial periodic pattern of P-positions in the main heaps.
If the gadget is out of phase

The gadget contains exactly one match, but this match does not correspond to the number of tokens in the time-heap modulo $k$. 
If the gadget is out of phase

a constant row

time heap’s 0
gadget’s 0

tape all P’s

**Figure:** The gadget “thinks” the time heap is congruent to zero when it is not. The gray moves will not be possible.
If the gadget is out of phase

- The least row that corresponds to a finished computation congruent to 0 modulo $k$ will be constant, since no information has been propagated sideways up to this point.
If the gadget is out of phase

- The least row that corresponds to a finished computation congruent to 0 modulo \( k \) will be constant, since no information has been propagated sideways up to this point.

- If constant 0 (N-positions), then the following pattern will be periodic, since the computation now restarts as if it had started on a tape of only zeros.
If the gadget is out of phase

- The least row that corresponds to a finished computation congruent to 0 modulo $k$ will be constant, since no information has been propagated sideways up to this point.
- If constant 0 (N-positions), then the following pattern will be periodic, since the computation now restarts as if it had started on a tape of only zeros.
- If constant 1 (P-positions), then the following behavior will be the same as if the gadget was in phase, since the computation now starts from a row of $\cdots 000111 \cdots$
Conclusion

- We have shown that a game consisting of two main heaps and a “gadget” can emulate any one-dimensional cellular automaton, and thereby any Turing machine.
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- The halting problem for a Turing machine is undecidable.
- Hence it is undecidable whether a given finite pattern of P-position occurs in our invariant heap game.
The difference of any two P-positions (equal Grundy values) in a game is impossible as a move in this game (in a disjunctive sum).
Game equivalence?

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- Two distinct P-positions constitute a finite pattern...
Further questions?

- How many heaps are required for undecidability? (strictly speaking we didn’t prove that any finite number of heaps leads to undecidability).
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- We guess that three heaps suffice, and perhaps even two.
- Do we need to be able to add tokens to heaps in order to achieve undecidability?