Impartial games on random graphs

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Idea:
Given some fixed distribution, generate a (possibly infinite) graph at random. The expected number of edges per node depends on some parameter.
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Play a coin-sliding game on this random graph.
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- What is the probability of a second player win?
- What is the probability of a draw?
The probability of non-loss and win of GWUG(\(\lambda\))
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- Geography, a well-known 2-player children game.
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The move rules
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The possibility of a draw game

We play normal play. A player who cannot move loses.
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The player not moving from $\nu$ wins
The player moving from $\nu$ wins
The player moving from $\nu$ wins
The Erdös-Rényi (ER) model

Let $n \in \mathbb{N}$ and $p \in [0, 1]$. Let $G(n, p)$ denote an ER-random graph on $n$ nodes where an edge $\{x, y\}$ is present with probability $p$. 
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Let $n \in \mathbb{N}$ and $p \in [0, 1]$. Let $G(n, p)$ denote an ER-random graph on $n$ nodes where an edge $\{x, y\}$ is present with probability $p$.

A Poissonian degree distribution sequence

The degree of a node is a binomially distributed random variable, $D$.

The expectation of $D$ is $(n-1)p$. Keeping the expected degree constant as $n \to \infty$, $D$ may be approximated with a Poissonian random variable with $\lambda = (n-1)p$ and so $\Pr(D = k) = \frac{\lambda^k}{k!} e^{-\lambda}$.

Threshold at $\lambda = 1$

If $\lambda > 1$, almost surely $G(n, p)$ contains one giant component of size $\Theta(n)$.

If $\lambda < 1$, the size of the largest connected component is $\Theta(\log(n))$.

The number of small cycles in a small component is small. Thus, locally, the graph resembles a Branching process.
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An instance of $G(100,0.01)$
A Galton Watson Branching process (GW)

Start with a single node $\nu$ at generation 0 and some fixed offspring distribution. Put $a_k = \Pr(D = k)$. 

Generating function $f(x) = \sum_{k=1}^{\infty} a_k x^k$.

$f(x)$ is monotone increasing.

The expected number of children per node is $f'(1)$.

If $f'(1) > 1$, the fixpoint equation $f(x) = x$ has two solutions.

$\Pr(\text{a branching process dies at generation 0}) = a_0$.

$\Pr(\text{a branching process has at most } k \text{ generations}) = f_k(a_0)$.

The probability of extinction is $f_\infty(a_0)$.
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- $\Pr(\text{a branching process dies at generation 0}) = a_0$.
- $\Pr(\text{a branching process has at most } k \text{ generations}) = f^k(a_0)$.
- The probability of extinction is $f^\infty(a_0)$. 
Survival and extinction

If $0 \leq f'(1) \leq 1$, the probability of extinction is 1.

If $f'(1) > 1$ there is a positive probability of survival, say $1 - \alpha$.

This probability is given by the least positive solution to the fixpoint equation: $f(\alpha) = \alpha$. 
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A blocking maneuver: $k$-blocking UVG

$f_{1.7}(x)$
Iterating $x_{k+1} = f_{1.7}(x_k)$
Here $a_k$ is given by a Poisson distribution with some fixed parameter $\lambda$, so that
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$$f(x) = f_\lambda(x) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda(1-x)},$$

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Putting $x_0 = a_0 > 0$ we may evaluate $\alpha = \lim_{k \to \infty} x_k$, where
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Putting $x_0 = a_0 > 0$ we may evaluate $\alpha = \lim_{k \to \infty} x_k$, where

$$x_{k+1} = f(x_k) = e^{-\lambda(1-x_k)}.$$
The previous player is the player not in turn to move. Put
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\[ p = \Pr(\text{The previous player wins}), \]
The game of GWUVG

The previous player is the player not in turn to move. Put

- \( p = \Pr(\text{The previous player wins}) \),
- \( q = \Pr(\text{The previous player does not lose}) \),

\[ \lim_{k \to \infty} p_k \to p \quad \text{and} \quad \lim_{k \to \infty} q_k \to q. \]
The previous player is the player not in turn to move. Put

- \( p = \Pr(\text{The previous player wins}) \),
- \( q = \Pr(\text{The previous player does not lose}) \),
- \( p_k = \Pr(\text{The previous player wins within } k \text{ generations}) \).
The previous player is the player not in turn to move. Put

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- $p_k = \Pr(\text{The previous player wins within } k \text{ generations})$,
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Then $\lim p_k \rightarrow p$ and $\lim q_k \rightarrow q$. 

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Impartial games on random graphs
The previous player is the player not in turn to move. Put

- $p = \Pr(\text{The previous player wins})$,
- $q = \Pr(\text{The previous player does not lose})$,
- $p_k = \Pr(\text{The previous player wins within $k$ generations})$,
- $q_k = \Pr(\text{The previous player does not lose within $k$ generations})$,
- Then $\lim p_k \to p$ and $\lim q_k \to q$. 
The initial conditions

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- cannot win before game has started: $p_0 = 0$. 
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- The previous player cannot win before game has started: $p_0 = 0$.
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The initial conditions

The previous player

- cannot win before game has started: $p_0 = 0$.
- cannot lose before game has started $q_0 = 1$.
- wins if there is no offspring: $p_1 = a_0 = e^{-\lambda}$.
The initial conditions

The previous player

- cannot win before game has started: \( p_0 = 0 \).
- cannot lose before game has started \( q_0 = 1 \).
- wins if there is no offspring: \( p_1 = a_0 = e^{-\lambda} \).
- cannot lose in the first generation since it is the first players turn: \( q_1 = q_0 = 1 \).
Looking below the first generation

Fix some distribution and let the root (generation 0) of a GW-tree serve as a starting position of UVG. Player A begins. Then

\[ p_{k+1} = \Pr(\text{Player B wins within the first } k+1 \text{ generations}) = \Pr(\text{Player A loses within the next } k \text{ generations}) = \Pr(\text{From each first generation child, the previous player loses within the next } k \text{ generations}) = \sum_{i=0}^{\infty} a_i (1 - q_i k)^i = f(1 - qk). \]
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Similarly:
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\[ q_{k+1} = \Pr(\text{Player B does not lose within the first } k + 1 \text{ generations}) \]
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\[ = \Pr(\text{From each first generation child, the previous player does not win within the next } k \text{ generations}) \]
\[ = \sum_{i=0}^{\infty} a_i (1 - p_k)^i = f(1 - p_k) = \]
$x$ and $e^{-2x}$
Iterating $p_k = e^{-2q_k}$ and $q_k = e^{-2p_k}$
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$x$ and $e^{-3.5x}$
Iterating $p_k = e^{-3.5q_k}$ and $q_k = e^{-3.5p_k}$
One-dimensional non-linear dynamics

Since $|f'_2(\alpha)| < 1$, the first fixpoint is an attractor.
One-dimensional non-linear dynamics

- Since $|f_2'(\alpha)| < 1$, the first fixpoint is an attractor.
- The second fixpoint is repellent, by $|f_{3.5}'(\alpha)| > 1$. 

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One-dimensional non-linear dynamics

- Since $|f'_2(\alpha)| < 1$, the first fixpoint is an attractor.
- The second fixpoint is repellent, by $|f'_{3.5}(\alpha)| > 1$.
- But it is an attractor of period 2:
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$x$ and $e^{-3.5e^{-3.5x}}$

This fixpoint is an attractor
since $|f(x^*)| < 1$

This fixpoint is no attractor
since $|f(x^*)| > 1$
A bifurcation at $\lambda = e$
A game theoretical interpretation

First player wins

Second player wins

Draw

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Impartial games on random graphs
\[ p < q \text{ if } \lambda > e \]

- \( q = 0.82 \)
- \( s = 0.24 \)
- \( p = 0.06 \)
- \( s = 3.5 \)
\[ p = q \text{ if } \lambda \leq e \]
Why a bifurcation at $\lambda = e$?

**Theorem**

*The probability for draw of UVG on a Poissonian GW-tree is 0 if and only if $\lambda \leq e$.***
Why a bifurcation at $\lambda = e$?

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The probability for draw of UVG on a Poissonian GW-tree is 0 if and only if $\lambda \leq e$.

**Proof.** Put $g(x) = f(1 - x)$. By one-dimensional non-linear dynamics, the fixpoint, say $g(\alpha) = \alpha$, is an attractor if and only if $|g'(\alpha)| \leq 1$. We get
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- $g(x) = e^{-\lambda x}$;
- $g'(x) = -\lambda e^{-\lambda x}$;
- $g'(x) < 0$. So $\alpha$ is an attractor if and only if $g'(\alpha) \geq -1$;
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- $g(x) = e^{-\lambda x}$;
- $g'(x) = -\lambda e^{-\lambda x}$;
- For all $x$, $g'(x) < 0$. So $\alpha$ is an attractor if and only if $g'(\alpha) \geq -1$;
- But $g'(\alpha) = -\lambda e^{-\lambda \alpha} = -\lambda \alpha \geq -1$;
Why a bifurcation at $\lambda = e$?

**Theorem**

The probability for draw of UVG on a Poissonian GW-tree is 0 if and only if $\lambda \leq e$.

**Proof.** Put $g(x) = f(1 - x)$. By one-dimensional non-linear dynamics, the fixpoint, say $g(\alpha) = \alpha$, is an attractor if and only if $|g'(\alpha)| \leq 1$. We get

- $g(x) = e^{-\lambda x}$;
- $g'(x) = -\lambda e^{-\lambda x}$;
- For all $x$, $g'(x) < 0$. So $\alpha$ is an attractor if and only if $g'(\alpha) \geq -1$;
- But $g'(\alpha) = -\lambda e^{-\lambda \alpha} = -\lambda \alpha \geq -1$;
- This gives $\lambda \leq e$. At the critical intensity the probability for a second player win is $\alpha = \frac{1}{e}$.

Urban Larsson, joint work with Johan Wästlund

Impartial games on random graphs
When does the first player win?

The intensity $s = e$ maximizes the probability $1 - p$ that the first player has a winning strategy.

$1 - p = 1 - e^t$
The expected size of a maximum matching in $G(n, p)$

The Karp-Sipser (1981) leaf removal algorithm on $G(n, p)$ gives a core that covers a finite fraction of all the vertices if

$$\lambda = (n - 1)p > e.$$
The expected size of a maximum matching in $G(n, p)$

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The Karp-Sipser (1981) leaf removal algorithm on $G(n, p)$ gives a core that covers a finite fraction of all the vertices if $\lambda = (n - 1)p > e$. If $\lambda \leq e$, asymptotically it does not cover any vertices. For large $n$, if the core is large all its nodes can be matched.
Suppose we play a game of UVG on a finite graph with $n$ nodes. Then, if no player can force a win within $\sqrt{\log(n)}$ moves, we define the outcome of the game as a pseudo-draw.
Suppose we play a game of UVG on a finite graph with $n$ nodes. Then, if no player can force a win within $\sqrt{\log(n)}$ moves, we define the outcome of the game as a pseudo-draw.

**Theorem**

*The probability for a pseudo-draw of UVG on $G(n, p)$ is 0 if and only if $\lambda \leq e$.***
A blocking maneuver

Definition
Let $k \in \mathbb{N}$. The rules of $k$-blocking UVG are as UVG with the following twist: Before the next player moves, the previous player may block off at most $k - 1$ edges and declare them as non-slidable.
A blocking maneuver

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Let $k \in \mathbb{N}$. The rules of $k$-blocking UVG are as UVG with the following twist: Before the next player moves, the previous player may block off at most $k - 1$ edges and declare them as non-slidable.

So 1-blocking UVG = UVG.
Looking below the first generation
Looking below the first generation

\[ p_{k+1} = \Pr(\text{Player B wins within the first } k + 1 \text{ generations}) \]
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\[ \rightarrow f(1 - p) + pf'(1 - p). \]
Hence, for 2-blocking UVG, if $a_i$ is Poissonian, we get:

$$q = (1 + \lambda p)e^{-\lambda p}$$

and

$$p = (1 + \lambda q)e^{-\lambda q}$$

and so for this game the critical intensity $\lambda_0 = \frac{e\phi}{\phi}$, where $\phi = \frac{1+\sqrt{5}}{2}$. 
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and so for this game the critical intensity $\lambda_0 = \frac{e^\phi}{\phi}$, where $\phi = \frac{1+\sqrt{5}}{2}$. At this intensity and below, the probability for a draw is 0. The probability for a player B win at this intensity is $\frac{\phi^2}{e^\phi}$. 
In general

Let \( k \in \mathbb{N} \). We summarize a generalization
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- A maximal partial $k$-Factor, $F$, provides a non-losing strategy for $k$-UVG on a rooted tree;
- A non-losing strategy is to slide along edges in $F$.
- Denote with $x_0$ the unique positive real root of the equation
  \[
  x^{k+1} = \frac{k!}{k!}x^k + \frac{k!}{(k-1)!}x^k + \ldots + \frac{k!}{1!}x + k!.
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In general

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- A maximal partial \( k \)-Factor, \( F \), provides a non-losing strategy for \( k \)-UVG on a rooted tree;
- A non-losing strategy is to slide along edges in \( F \).
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  x^{k+1} = \frac{k!}{k!} x^k + \frac{k!}{(k-1)!} x^k + \ldots + \frac{k!}{1!} x + k!.
  \]
- The critical intensity for \( k \)-blocking GWUVG is \( \lambda_0 = \frac{k! e^{x_0}}{x_0^k} \).

The probability for a Second player win is \( \alpha = \frac{x_0^{k+1}}{k! e^{x_0}} \).
Other distributions?

Let $a_i$ be uniformly distributed on $0, 1, \ldots, N-1$ so that $a_i = 1/N$ if $i \in \{0, 1, \ldots, N-1\}$, and zero otherwise. Denote UVG on this GW process $N$-GW.

**Theorem**

*The probability for a draw on $N$-GW with uniform distribution is zero for all $N \geq 0$. For $N = 2, 3$ the second player wins with probability $2/3$ and $3 - \sqrt{6} \approx 0.55$. For $N > 3$ the probability for a first player win is $> 0.5$.*
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Is this the end of the story of random 'bifurcation games'?
Wighted Heads (= 0 children) and tails (= 2)?

\[
t = \Pr(\text{a node has precisely 2 children})
\]

\[
1 - t = \Pr(\text{a node has no children})
\]

\[
f(x) = 1 - t + tx^2
\]

\[
|f(x^*)| < 1
\]

\[
|f(x^*)| > 1
\]

\[
t = 1 - 1/\sqrt{3} \approx 0.866
\]

First player wins

Second player wins

Draw

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