RESTRICTIONS OF m-WYTHOFF NIM AND p-COMPLEMENTARY BEATTY SEQUENCES

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Abstract. Fix a positive integer m. The game of *m-Wythoff Nim* (A.S. Fraenkel, 1982) is a well-known extension of *Wythoff Nim*, a.k.a 'Corner the Queen'. Its set of P-positions may be represented by a pair of increasing sequences of non-negative integers. It is well-known that these sequences are so-called *complementary homogeneous Beatty sequences*, that is they satisfy Beatty's theorem. For a positive integer p, we generalize the solution of m-Wythoff Nim to a pair of *p-complementary*—each positive integer occurs exactly p times—homogeneous Beatty sequences $a = (a_n)_{n\in\mathbb{Z}_{\geq 0}}$ and $b = (b_n)_{n\in\mathbb{Z}_{\geq 0}}$, which, for all n, satisfies $b_n - a_n = mn$. By the latter property, we show that a and b are unique among *all* pairs of non-decreasing p-complementary sequences. We prove that such pairs can be partitioned into p pairs of complementary Beatty sequences. Our main results are that $\{\{a_n, b_n\} \mid n \in \mathbb{Z}_{\geq 0}\}$ represents the solution to three new 'p-restrictions' of m-Wythoff Nim—of which one has a *blocking maneuver* on the *rook-type* options. C. Kimberling has shown that the solution of Wythoff Nim satisfies the *complementary equation* $x_{x_n} = y_n - 1$. We generalize this formula to a certain 'p-complementary equation' satisfied by our pair a and b. We also show that one may obtain our new pair of sequences by three so-called *Minimal EXclusive* algorithms. We conclude with an Appendix by Aviezri Fraenkel.

1. Introduction and notation

The combinatorial game of *Wythoff Nim* ([Wy07]) is a so-called (2-player) impartial game played on two piles of tokens. (For an introduction to impartial games see [ANW07, BCG82, C76].) As an addition to the rules of the game of Nim ([Bo02]), where the players alternate in removing any finite number of tokens from precisely one of the piles (at most the whole pile), Wythoff Nim also allows removal of the same number of tokens from both piles. The player who removes the last token wins.

This game is more known as 'Corner the Queen', invented by R. P. Isaacs (1960), because the game can be played on a (large) Chess board with one single Queen. Two players move the Queen alternately but with the restriction that, for each move, the (L^1) distance to the lower left corner, position $(0, 0)$, must decrease. (The Queen must at all times remain on the board.) The player who moves to this *final/terminal* position wins.

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In this paper we follow the convention to denote our players with the *next player* (the player who is in turn to move) and the *previous player*. A *P-position* is a position from which the previous player can win (given perfect play). An *N-position* is a position from which the next player can win. Any position is either a P-position or an N-position. We denote *the solution*, the set of all P-positions, of an impartial game G, by $\mathcal{P} = \mathcal{P}(G)$ and the set of all N-positions by $\mathcal{N} = \mathcal{N}(G)$. The positive integers are denoted by $\mathbb{Z}_{>0}$ and the non-negative integers by $\mathbb{Z}_{\geq 0}$. Let $x = (x_i)$ denote an integer sequence over some index set and let $\xi \in \mathbb{Z}$. Then, we let $x_{>\xi}$ denote $(x_i)_{i>\xi}$.

1.1. **Restrictions of m-Wythoff Nim.** Let $m \in \mathbb{Z}_{>0}$. We next turn to a certain m-extension of Wythoff Nim, studied in [Fr82] by A.S. Fraenkel. In the game of *m-Wythoff Nim*, or just mWN (our notation), the Queen's 'bishop-type' options are extended so that $(x+i, y+j) \to (x, y)$ is legal if $|\, i-j \,| < m$, $i, j, x, y \in \mathbb{Z}_{\geq 0}$, $i > 0$ or $j > 0$. The *rook-type* options are as in Nim. Hence 1-Wythoff Nim is identical to Wythoff Nim.

In this paper we define three new *restrictions* of m-Wythoff Nim—here a rough outline:

- The first has a so-called *blocking maneuver/Muller Twist* (see also [HR, SS02] and Section 1.6 of this paper) on the rook-type options, before the next player moves, the previous player may announce some of these options as forbidden;
- The second has a certain congruence restriction on the rook-type options;
- For the third, a rectangular piece is removed from the lower left corner of the game board (including position $(0, 0)$).

Depending on the particular setup, in addition to $(0, 0)$, the first two games may have a finite number of final positions of the form $\{0, x\}$[1], $x \in \mathbb{Z}_{>0}$. In the third game, there are precisely two final positions $\neq (0, 0)$.

1.2. **Beatty sequences and p-complementarity.** A (general) *Beatty sequence* is a non-decreasing integer sequence of the form

$$(\lfloor n\alpha + \gamma \rfloor), \tag{1}$$

usually indexed over $\mathbb{Z}, \mathbb{Z}_{\geq 0}$ or $\mathbb{Z}_{>0}$, where α is a positive irrational and $\gamma \in \mathbb{R}$. S. Beatty [Be26] is probably most known for a (re)[2]discovery of (the statement of) the following theorem: If α and β are positive real numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ then $(\lfloor n\alpha \rfloor)_{\mathbb{Z}_{>0}}$ and $(\lfloor n\beta \rfloor)_{\mathbb{Z}_{>0}}$ partition $\mathbb{Z}_{>0}$ if and only if they are Beatty sequences. This was proved in [HO27] (see also [Fr82]). A pair of sequences that partition $\mathbb{Z}_{>0}$ ($\mathbb{Z}_{\geq 0}$, $\mathbb{Z}$) is usually called *complementary* (see [Fr69, Fr73, Ki07, Ki08]). Let us generalize this notion.

Definition 1. Let $p \in \mathbb{Z}_{>0}$ and $Q, R, S \subset \mathbb{Z}$. Two sequences $(x_i)_{i\in Q}$ and $(y_i)_{i\in R}$ of non-negative integers are *p-complementary* (on S), if, for each $n \in S$,

$$\#\{i \in Q \mid x_i = n\} + \#\{i \in R \mid y_i = n\} = p.$$

[1] For integers $0 \leq x \leq y$ we use the notation $\{x, y\}$ whenever (x, y) and (y, x) are considered the same.

[2] This theorem was in fact discovered by J. W. Rayleigh, see [Ra94, O'B03].

Remark 1. Since the main topics in this paper are three games mostly played on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, we often find it convenient to use $S = \mathbb{Z}_{\geq 0}$ in Definition 1. Also, for our purposes, it will be convenient use $\mathbb{Z}_{\geq 0}$ or $\mathbb{Z}_{>0}$ as the index sets R and Q. In the Appendix Aviezri Fraenkel provides an alternative approach.

We study the Beatty sequences $a = (a_n)_{n \in \mathbb{Z}_{\geq 0}}$ and $b = (b_n)_{n \in \mathbb{Z}_{\geq 0}}$, where for all $n \in \mathbb{Z}_{\geq 0}$,

$$a_n = a_n^{m,p} = \left\lfloor \frac{n\phi_{(mp)}}{p} \right\rfloor \tag{2}$$

and

$$b_n = b_n^{m,p} = \left\lfloor \frac{n(\phi_{(mp)} + mp)}{p} \right\rfloor, \tag{3}$$

and where

$$\phi_k = \frac{2 - k + \sqrt{k^2 + 4}}{2}. \tag{4}$$

We show that a and $b_{>0}$ are p-complementary.

In [Wy07] W.A. Wythoff proved that the solution of Wythoff Nim is given by $\{\{a_n^{1,1}, b_n^{1,1}\} \mid n \in \mathbb{Z}_{\geq 0}\}$. Then in [Fr82] it was shown that the solution of m-Wythoff Nim is

$$\{\{a_n^{m,1}, b_n^{m,1}\} \mid n \in \mathbb{Z}_{\geq 0}\}.$$

1.3. **Recurrence.** Let X be a strict subset of the non-negative integers. Then the *Minimal EXclusive*, for short MEX, of X is defined as usual (see [C76]):

$$\operatorname{mex} X := \min(\mathbb{Z}_{\geq 0} \setminus X).$$

For $n \in \mathbb{Z}_{\geq 0}$ put

$$x_n = \operatorname{mex}\{x_i, y_i \mid i \in \{0, 1, \ldots, n-1]\} \text{ and } y_n = x_n + mn. \tag{5}$$

With notation as in (5), it was proved in [Fr82] that $(x_n) = (a_n^{m,1})$ and $(y_n) = (b_n^{m,1})$. The MEX-algorithm in (5) gives an exponential time solution to mWN whereas the Beatty-pair in (2) and (3) give a polynomial time ditto. (For interesting discussions on complexity issues for combinatorial games, see for example [Fr04, FP09].) We show that one may, for general m and p, obtain a and b by three MEX-algorithms, which in various ways generalize (5).

It is well-known that the solution of Wythoff Nim satisfies the *complementary equation* (see for example [Ki95, Ki07, Ki08])

$$x_{x_n} = y_n - 1.$$

For arbitrary positive integers m and p, we generalize this formula to a 'p-complementary equation'

$$x_{\varphi_n} = y_n - 1, \tag{6}$$

where $\varphi_n := \frac{x_n + (mp-1)y_n}{m}$ (or $\varphi_n := py_n - n$), and show that a solution is given by $x = a$ and $y = b$.

1.4. **I.G. Connell's restriction of Wythoff Nim.** In the literature there is another generalization of Wythoff Nim that is of special interest to us. Let $p \in \mathbb{Z}_{>0}$. In [Co59] I.G. Connell studies the restriction of Wythoff Nim, where the the rook-type options are restricted to jumps of precise multiples of p. This game we call Wythoff modulo-p Nim and denote with $\mathrm{WN}^{(p)}$. Hence Wythoff modulo-1 Nim equals Wythoff Nim.

From [Co59] one may derive that $\mathcal{P}(\mathrm{WN}^{(p)}) = \{\{a_n^{1,p}, b_n^{1,p}\} \mid n \in \mathbb{Z}_{\geq 0}\}$.

$b_n^{1,3}$	0	1	2	4	5	7	8	10	11	12	14	15	17	18	20	21	22
$a_n^{1,3}$	0	0	0	1	1	2	2	3	3	3	4	4	5	5	6	6	6
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

TABLE 1. The initial P-positions of Connell's restriction of Wythoff Nim, $\mathrm{WN}^{(3)}$.

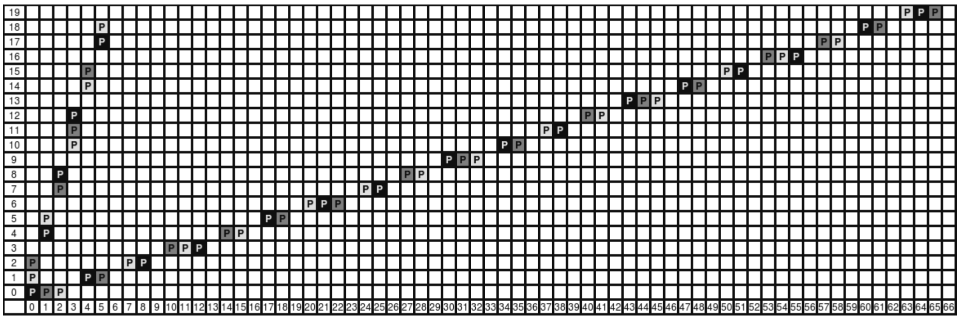

FIGURE 1. The P-positions of $\mathrm{WN}^{(3)}$ are the positions nearest the origin such that there are precisely three positions in each row and column and one position in each NE-SW-diagonal. The black positions represent the first few P-positions of 3-Wythoff Nim, namely the positions nearest the origin such that there is precisely one position in each row and one position in every third NE-SW diagonal. Positions with lighter shades represent the solutions of games in our third variation.

Remark 2. In Connell's presentation, for the proof of the above formulas, he uses p pairs of complementary sequences of integers (in analogy with the discovery of a new formulation of Beatty's theorem in [Sk57]). We have indicated this pattern of P-positions with different shades in Figure 1. In fact, the black positions, starting with $(0, 0)$ are P-positions of 3-Wythoff Nim. More generally, it is immediate by (2) and (3) that, for all p and n, $a_n^{p,1} = a_{pn}^{1,p}$ and $b_n^{p,1} = b_{pn}^{1,p}$.

Remark 3. In [BoFr73], Fraenkel and I. Borosh study yet another variation of both m-Wythoff Nim and Wythoff modulo-p Nim which includes a (different from ours) Beatty-type characterization of the P-positions.

1.5. **Exposition.** In Section 2, given fixed game constants $m, p \in \mathbb{Z}_{>0}$, we define our games, exemplify them and state our Main Theorem. Roughly: For each of our games, a position is P if and only if it is of the form $\{a_n^{m,p}, b_n^{m,p}\}$, with a and b as in (2) and (3) respectively. In Section 3 we generalize Beatty's theorem to pairs of p-complementary sequences, establish that such sequences can be partitioned into p pairs of complementary Beatty sequences and at last prove some arithmetic properties of a and b—most important of which is that a and $b_{>0}$ are p-complementary and, for all $n \in \mathbb{Z}_{\geq 0}$, satisfy $b_n = a_n + mn$. Then, in Section 4, we prove that the latter two properties make a and b unique among all pairs of non-decreasing sequences. Section 5 is devoted to equation (6) and three MEX-algorithms. In Section 6 we prove our game theory results (stated in Section 2) and in Section 7 a few questions are posed. At last there is an Appendix, provided by Aviezri Fraenkel.

Let us, before we move on to our games, give some more background to the so-called *blocking maneuver* in the context of Wythoff Nim.

1.6. **A bishop-type blocking variation of m-Wythoff Nim.** Let $m, p \in \mathbb{Z}_{>0}$. In [HL06] we gave an exponential time solution to a variation of m-Wythoff Nim with a 'bishop-type' blocking maneuver, denoted by p-Blocking m-Wythoff Nim (and with (m,p)-Wythoff Nim in [La09]).

The rules are as in m-Wythoff Nim, except that before the next player moves, the previous player is allowed to block off (at most) $p-1$ bishop-type—note, not m-bishop-type—options and declare that the next player must refrain from these options. When the next player has moved, any blocking maneuver is forgotten.

The solution of this game is in a certain sense 'very close' to pairs of Beatty sequences (see also the Appendix of [La09]) of the form

$$\left(\left\lfloor n\frac{\sqrt{m^2+4p^2}+2p-m}{2p}\right\rfloor\right) \text{ and } \left(\left\lfloor n\frac{\sqrt{m^2+4p^2}+2p+m}{2p}\right\rfloor\right).$$

But we explain why there can be no Beatty-type solution to this game for $p > 1$. However, in [La09], for the cases $p \,|\, m$, we give a certain 'Beatty-type' characterization. For these kind of questions, see also [BF84]. However, a recent discovery, in [Ha, FP09], provides a polynomial time algorithm for the solution of (m,p)-Wythoff Nim (for any combination of m and p).

An interesting connection to 4-Blocking 2-Wythoff Nim is presented in [DG08], where the authors give an explicit bijection of solutions to a variation of Wythoff's original game, where a player's bishop-type move is restricted to jumps by multiples of a predetermined positive integer.

For another variation, [La09] defines the rules of a so-called move-size dynamic variation of two-pile Nim, (m,p)-Imitation Nim, for which the P-positions, treated as starting positions, are identical to the P-positions of (m,p)-Wythoff Nim.

This discovery of a 'dual' game to (m,p)-Wythoff Nim has in its turn motivated the study of 'dual' constructions of the rook-type blocking maneuver in this paper.

2. Three games

This section is devoted to defining and exemplifying our game rules and stating our main results. We begin by introducing some (non-standard) notation whereby we decompose the Queen's moves into *rook-type* and *bishop-type* ditto.

Definition 2. Fix $m, p \in \mathbb{Z}_{>0}$ and an $l \in \{0, 1, \ldots, m\}$.

(i) An (l, p)-*rook* moves as in Nim, but the length of a move must be $ip + j > 0$ positions for some $i \in \mathbb{Z}_{\geq 0}$ and $j \in \{0, 1, \ldots, l-1\}$ (we denote a $(0, p)$-rook by a p-rook and a (p, p)-rook simply by a *rook*);

(ii) A m-*bishop* may move $0 \leq i < m$ rook-type positions and then any number of, say $j \geq 0$, bishop-type positions (a *bishop* moves as in Chess), all in one and the same move, provided $i + j > 0$ and the L^1-distance to $(0, 0)$ decreases.

2.1. **Game definitions.** As is clear from Definition 2 the rook-type options intersect the m-bishop-type options precisely when $m > 1$. For example, $(2, 3) \to (1, 3)$ is both a 2-bishop-type and a rook-type option. We will make use of this fact when defining the blocking maneuver. Therefore, let us introduce some new terminology.

Definition 3. Fix $m \in \mathbb{Z}_{>0}$. A rook-type option, which is not of the form of the m-bishop as in Definition 2 (ii), is a *roob-type*[3] option.

Hence, for $m = 2$, a *roob* may move $(2, 3) \to (2, 1)$, but not $(2, 3) \to (2, 2)$ (both are rook-type options). Let us define our games.

Definition 4. Fix $m, p \in \mathbb{Z}_{>0}$.

(i) The game of m-*Wythoff* p-*Blocking Nim*, or $m\mathrm{WN}^p$, is a restriction of m-Wythoff Nim with a roob-type blocking maneuver. The Queen moves as in m-Wythoff Nim (that is, as the m-bishop or the rook), but with one exception: Before the next player moves, the previous player may *block off* (at most) $p - 1$ of the next player's roob-type options. Any blocked option is unavailable for the next player. As usual, each blocking maneuver is particular to a specific move; that is, when the next player has moved, any blocking maneuver is forgotten and has no further impact on the game. (For $p = 1$ this game equals m-Wythoff Nim.)

(ii) Fix an integer $0 \leq l < p$. In the game of m-*Wythoff Modulo*-p l-*Nim*, or $m\mathrm{WN}^{(l,p)}$, the Queen moves as the m-bishop or the (l, p)-rook. For $l = 0$ we denote this game by m-*Wythoff Modulo*-p *Nim* or $m\mathrm{WN}^{(p)}$. (In case $l = p$ the game is simply m-Wythoff Nim.)

(iiia) Fix an integer $0 \leq l < p$. In the game of l-*Shifted* $m{\times}p$-*Wythoff Nim*, or $m{\times}p\mathrm{WN}_l$, the Queen moves as in (mp)-Wythoff Nim (that is, as the (mp)-bishop or the rook), except that, if $l > 0$, it is not allowed to move to a position of the form (i, j), where $0 \leq i < ml$ and $0 \leq j < m(p-l)$. Hence, for this case, the terminal positions are

[3] Think of 'roob' as 'ROOk minus m-Bishop', or maybe 'ROOk Blocking'

$(ml, 0)$ and $(0, m(p-l))$.[4] On the other hand $m{\times}p\mathrm{WN}_0$ is identical to (mp)-Wythoff Nim.

(iiib) The game of $m{\times}p$-Wythoff Nim, $m{\times}p$WN: Before the first player moves, the second player may decide the parameter l as in (3a). Once the parameter l is fixed, it remains the same until the game has terminated, so that for the remainder of the game, the rules are as in $m{\times}p\mathrm{WN}_l$.

2.2. **Examples.** Let us illustrate some of our games, where our players are *Alice* and *Bob*—Alice makes the first move (and Bob makes the first blocking maneuver in case the game has a Muller twist).

Example 1. Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^2$. Then the only bishop-type move is $(0, 2) \to (0, 1)$. There is precisely one roob-type option, namely $(0, 0)$. Since this is a terminal position Bob will block it off from Alice's options, so that Alice has to move to $(0, 1)$. The move $(0, 1) \to (0, 0)$ cannot be blocked off for the same reason, so Bob wins. If $y \geq 3$ there is always a move $(0, y) \to (0, x)$, where $x = 0$ or 2. This is because the previous player may block off at most one option. Altogether, this gives that $\{0, y\}$ is P if and only if $y = 0$ or 2.

Example 2. Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^{(2)}$. Alice can move to $(0, 0)$, since $0 \equiv 2 \pmod 2$, so $(0, 2)$ is N. On the other hand, the position $(0, 3)$ is P since the only options are $(0, 2)$ and $(0, 1)$. (The latter is N since the 2-bishop can move $(0, 1) \to (0, 0)$.)

Example 3. Suppose the starting position is $(0, 2)$ and the game is $2\mathrm{WN}^{(2,4)}$. Alice cannot move to a P-position since $2 - 0 \not\equiv 0, 1 \pmod 4$ and the 2-bishop's move is restricted to $(0, 1)$, which is N. Hence $(0, 2)$ is P. More generally, $(0, y)$ is N for all $y \geq 3$, since $(0, y) \to (0, 0)$ is legal if $0 < y \equiv 0, 1 \pmod 4$ and $(0, y) \to (0, 2)$ is legal if $2 < y \equiv 2, 3 \pmod 4$.

Example 4. Suppose the starting position is $(0, 4)$ and the game is $2\mathrm{WN}^3$. Then the only bishop-type move is $(0, 4) \to (0, 3)$, so that the roob-type options are $(0, 0), (0, 1), (0, 2)$. Bob may block off 2 of these positions, say $(0, 0), (0, 2)$. Then if Alice moves to $(0, 1)$ she will loose (since she may not block off (0,0)), so suppose rather that she moves to $(0, 3)$. Than she may not block off $(0, 2)$ so Bob moves $(0, 3) \to (0, 2)$ and blocks off $(0, 0)$. Hence $(0, 4)$ is a P-position.

Example 5. Suppose the starting position is $(0, 4)$ and the game is $2\mathrm{WN}^{(3)}$. Alice cannot move to $(0, 0)$ or $(0, 2)$. But $(0, 1) \to (0, 0)$ is a 2-bishop-type option and $(0, 3) \to (0, 0)$ is a 3-rook-type option. This shows that $(0, 4)$ is a P-position.

[4] One might want to think of the game board as if a rectangle with circumference $2mp$ is cut out from its lower left corner. By symmetry, there are $\lceil \frac{p}{2} \rceil$ rectangle shapes but (given a starting position) p distinct game boards, see also (3b). Of course, if cutting out a corner of the nice game board does not appeal to the players, one might just as well define all positions inside the rectangle as N. Notice the close relation of these games to the misère version of m-Wythoff Nim studied in [Fr84].

Notice that, in comparison to Examples 4 and 5, the P-positions in the Examples 1 and 2 are distinct in spite the identical game constants ($m = p = 2$). On the other hand, the P-positions in Examples 1 and 3 coincide.

Example 6. Suppose the game is $2{\times}3\mathrm{WN}_1$, then the terminal positions are $(2,0)$ and $(0,4)$. On the other hand, for the game $2{\times}3\mathrm{WN}_2$, the positions $(0,2)$ and $(4,0)$ are terminal. Suppose now that the starting position of $2{\times}3\mathrm{WN}_2$ is $(1,9)$. Then Alice wins by moving to $(0,4)$. If the starting position is the same, but the game is $2{\times}3\mathrm{WN}_1$, then Alice cannot move to $(0,2)$ and hence Bob wins. Similarly, if the starting position of $2{\times}3\mathrm{WN}_0$ is $(1,7)$ Alice may not move to $(0,0)$ and hence Bob wins.

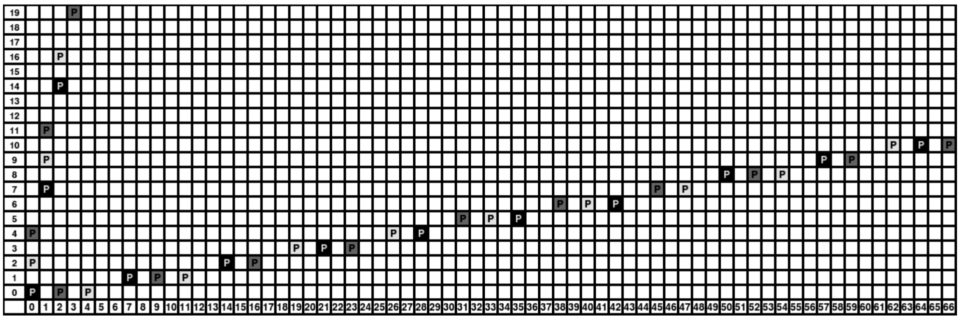

FIGURE 2. P-positions of $2\mathrm{WN}^{(3)}$, $2\mathrm{WN}^{3}$, $2\mathrm{WN}^{2,6}$ and $2\times$ 3WN—the positions nearest the origin such that there are precisely three positions in each row and column and one position in every second NE-SW-diagonal. The palest colored squares represent P-positions of $2{\times}3\mathrm{WN}_1$. They are of the form (a_{3n+1}, b_{3n+1}) or (b_{3n+2}, a_{3n+2}). The darkest squares, $(\{a^{2,3}_{3i}, b^{2,3}_{3i}\})$, represent the solution of 6WN.

$b^{2,3}_n$	0	2	4	7	9	11	14	16	19	21	23	26	28	31	33	35	38
$a^{2,3}_n$	0	0	0	1	1	1	2	2	3	3	3	4	4	5	5	5	6
$b_n - a_n$	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

TABLE 2. Some initial values of the Beatty pairs defined in (2) and (3), here $m = 2$ and $p = 3$, together with the differences of their coordinates (=$2n$).

2.3. **Game theory results.** We may now state our main results. We prove them in Section 6, since our proofs depend on some arithmetic results presented in Section 3, 4 and 5.

Theorem 2.1 (Main Theorem)**.** Fix $m, p \in \mathbb{Z}_{>0}$ and let a and b be as in (2) and (3). Then

(i) $\mathcal{P}(m\mathrm{WN}^p) = \{\{a_i, b_i\} \mid i \in \mathbb{Z}_{\geq 0}\}$;

(ii) (a) $\mathcal{P}(m\text{WN}^{(p)}) = \{\{a_i, b_i\} \mid i \in \mathbb{Z}_{\geq 0}\}$ if and only if $\gcd(m, p) = 1$;
(b) $\mathcal{P}(m\text{WN}^{(m,mp)}) = \{\{a_i, b_i\} \mid i \in \mathbb{Z}_{\geq 0}\}$;

(iii) (a) $\mathcal{P}(m{\times}p\text{WN}_l) = \{(a_{ip+l}, b_{ip+l}) \mid i \in \mathbb{Z}_{\geq 0}\} \cup \{(b_{ip-l}, a_{ip-l}) \mid i \in \mathbb{Z}_{>0}\}$
(b) $\mathcal{P}(m{\times}p\text{WN}) = \{\{a_i, b_i\} \mid i \in \mathbb{Z}_{\geq 0}\}$.

By this result it is clear that, in terms of game complexity, the solution of each of our games is polynomial.

3. More on p-complementary Beatty sequences

As we have seen, it is customary to represent the solution of 'a removal game on two heaps of tokens' as a sequence of (ordered) pairs of non-negative integers. However, often it turns out that it is more convenient to study the corresponding pair of sequences of non-negative integers. Sometimes, as in Wythoff Nim, these sequences are increasing. It turns out that for our purpose we are more interested in pairs of non-decreasing sequences. This leads us to a certain extension of Beatty's original theorem, to pairs of p-complementary Beatty sequences.

In the literature there is a proof of this theorem in [O'B02], where K. O'Bryant uses generating functions (a method adapted from [BB93]). Here, we have chosen to include an elementary proof, in analogy to ideas presented in [HO27, Fr82]. (See also the Appendix.)

Theorem 3.1 (O'Bryant)**.** Let $p \in \mathbb{Z}_{>0}$ and let $0 < \alpha < \beta$ be real numbers such that

$$\frac{1}{\alpha} + \frac{1}{\beta} = p. \tag{7}$$

Then $(\lfloor i\alpha \rfloor)_{i \in \mathbb{Z}_{\geq 0}}$ and $(\lfloor i\beta \rfloor)_{i \in \mathbb{Z}_{>0}}$ are p-complementary on $\mathbb{Z}_{\geq 0}$ if and only if α, β are irrational.

Proof. It suffices to establish that exactly p members of the set

$$S = \{0, \alpha, \beta, 2\alpha, 2\beta, \ldots\}$$

is in the interval $[n, n+1)$ for each $n \in \mathbb{Z}_{\geq 0}$. But for any fixed n we have

$$\begin{aligned}\#(S \cap [0, n]) &= \#(\{0, \alpha, 2\alpha, \ldots\} \cap [0, n]) + \#(\{\beta, 2\beta, \ldots\} \cap [1, n]) \\ &= \lfloor n/\alpha \rfloor + 1 + \lfloor n/\beta \rfloor.\end{aligned}$$

But α and β are irrational if and only if, for all n,

$$\begin{aligned}np - 1 = n/\alpha + n/\beta - 1 &< \lfloor n/\alpha \rfloor + 1 + \lfloor n/\beta \rfloor \\ &< n/\alpha + n/\beta + 1 = np + 1.\end{aligned}$$

This gives $\lfloor n/\alpha \rfloor + 1 + \lfloor n/\beta \rfloor = np$. Going from n to $n+1$ gives the result. □

The following result establishes that a pair of p-complementary homogeneous Beatty sequences can always be partitioned into p complementary pairs of Beatty sequences. For the proof we use a generalization of Beatty's theorem in [Sk57, Fr69, O'B03].

Proposition 3.2. Let $2 \leq p \in \mathbb{Z}$. Suppose that $(x_i) = (\lfloor \alpha i \rfloor)_{i \in \mathbb{Z}_{\geq 0}}$ and $(y_i) = (\lfloor \beta i \rfloor)_{i \in \mathbb{Z}_{>0}}$ are p-complementary homogeneous Beatty sequences with $0 < \alpha < \beta$. Then, for any fixed integer $0 \leq l < p$, the pair of sequences

$$(x_{pi+l})_{i \in \mathbb{Z}_{\geq 0}} \text{ and } (y_{pi-l})_{i \in \mathbb{Z}_{>0}}$$

is complementary on $\mathbb{Z}_{\geq 0}$.

Proof. Since (x_i) and (y_i) are non-decreasing and p-complementary, we get that both (x_{pi+l}) and (y_{pi-l}) are increasing. Then, by $x_l = \min\{x_{pi+l}\}$ and $y_{p-l} = \min\{y_{pi-l}\}$, again, p-complementarity gives that

$$\max\{x_l, y_{p-l}\} > \min\{x_l, y_{p-l}\} = 0$$

and so, we may conclude that the integer 0 occurs precisely once together in (x_{pi+l}) and (y_{pi-l}). Let us adapt to the terminology in [O'B03].

Case $x_l = 0$: Then $l < \frac{1}{\alpha}$. For all $n \in \mathbb{Z}_{>0}$, we denote

$$\left\lfloor \frac{n - \gamma'}{\gamma} \right\rfloor = x_{pn+l}$$

and

$$\left\lfloor \frac{n - \eta'}{\eta} \right\rfloor = y_{pn-l}.$$

This gives $\gamma = \frac{1}{p\alpha}$, $\gamma' = -\frac{l}{p}$, $\eta = \frac{1}{p\beta}$ and $\eta' = \frac{l}{p}$. Then according to Fraenkel's Partition Theorem in [O'B03, page 5], since α is irrational, (x_{pn+l}) and (y_{pn-l}) are complementary on $\mathbb{Z}_{>0}$ if and only if

(i) $0 < \gamma < 1$,
(ii) $\gamma + \eta = 1$
(iii) $0 \leq \gamma + \gamma' \leq 1$
(iv) $\gamma' + \eta' = 0$ and $k\gamma + \gamma' \notin \mathbb{Z}$ for $2 \leq k \in \mathbb{Z}$.

These four items are easy to verify.
Item (i): This follows since (7) together with $0 < \alpha < \beta$ is equivalent to $\frac{1}{p} < \alpha < \frac{2}{p}$.
Item (ii): This is immediate by (7).
Item (iii): We have that $0 \leq \gamma + \gamma' \leq 1$ if and only if $0 \leq \frac{1}{p\alpha} - \frac{l}{p} \leq 1$ if and only if $l \leq \frac{1}{\alpha} \leq p + l$.
Item (iv): We have that $\eta' + \gamma' = \frac{l}{p} - \frac{l}{p} = 0$. Since γ is irrational and γ' is rational the latter part holds as well.

Case $y_{p-l} = 0$: Then $p - l < \frac{1}{\beta} = p - \frac{1}{\alpha}$, so that $l > \frac{1}{\alpha}$. For all $n \in \mathbb{Z}_{>0}$, we denote

$$\left\lfloor \frac{n - \gamma'}{\gamma} \right\rfloor = x_{p(n-1)+l}$$

and

$$\left\lfloor \frac{n - \eta'}{\eta} \right\rfloor = y_{p(n+1)-l}.$$

This gives $\gamma = \frac{1}{p\alpha}$, $\gamma' = \frac{p-l}{p}$, $\eta = \frac{1}{p\beta}$ and $\eta' = \frac{l-p}{p}$. Then items (i), (ii) and (iv) are treated in analogy with the first case. For (iii), since $\alpha l > 1$, we get

$0 \leq \gamma + \gamma' = \frac{1}{p\alpha} + \frac{p-l}{p} = 1 + \frac{1-\alpha l}{\alpha p} < 1.$ □

We will now focus on the properties of the sequences a and b. The next result is central to the rest of the paper.

Lemma 3.3. Fix $m, p \in \mathbb{Z}_{>0}$ and let a and b be as in (2) and (3) respectively. Then for each $n \in \mathbb{Z}_{\geq 0}$ we have that

(i) a and b are p-complementary;
(ii) $b_n - a_n = mn$;
(iii) if $p = 1$, then
  (a) $a_{n+1} - a_n = 1$ and $b_{n+1} - b_n = m + 1$, or
  (b) $a_{n+1} - a_n = 2$ and $b_{n+1} - b_n = m + 2$;
(iv) if $p > 1$, then
  (a) $a_{n+1} - a_n = 0$ and $b_{n+1} - b_n = m$, or
  (b) $a_{n+1} - a_n = 1$ and $b_{n+1} - b_n = m + 1$.

Proof. Since ϕ_x is irrational and $\frac{1}{\phi_x} + \frac{1}{\phi_x + x} = 1$, case (i) is immediate from Theorem 3.1.

For case (ii) put $\nu = \nu(m, p) = \frac{\phi_{mp}}{p} + \frac{m}{2}$ and observe that

$$b_n - a_n = \left\lfloor n\left(\nu + \frac{m}{2}\right)\right\rfloor - \left\lfloor n(\nu - \frac{m}{2})\right\rfloor.$$

If mn is even, we are done, so suppose that $mn - 1 = 2k$, $k \in \mathbb{Z}_{\geq 0}$. Then

$$b_n - a_n = \left\lfloor n\nu + \frac{1}{2}\right\rfloor - \left\lfloor n\nu - \frac{1}{2}\right\rfloor + 2k = 1 + 2k = mn.$$

For case (iii), by [Fr82], we are done. In case $p > 1$, by the triangle inequality, we get

$$\begin{aligned} 0 &< \frac{\phi_{mp}}{p} \\ &= \frac{1}{p} - \frac{m}{2} + \sqrt{\frac{m^2}{4} + \frac{1}{p^2}} \\ &< \frac{1}{p} + \frac{1}{p} \\ &\leq 1, \text{ since } p \geq 2, \end{aligned}$$

so that we may estimate

$$a_{n+1} - a_n = \left\lfloor \frac{(n+1)\phi_{mp}}{p} \right\rfloor - \left\lfloor \frac{n\phi_{mp}}{p} \right\rfloor \in \{0, 1\}.$$

Then by (ii) we have

$$\begin{aligned} b_{n+1} - b_n &= a_{n+1} + m(n+1) - a_n - mn \\ &= a_{n+1} - a_n + m, \end{aligned}$$

so that (iv) holds. □

4. A unique pair of p-complementary Beatty sequences

For fixed p and m we now present a certain uniqueness property for our pair of p-complementary Beatty sequences (in case $p = 1$ see also [HL06] for extensive generalizations).

Theorem 4.1. Fix $m, p \in \mathbb{Z}_{>0}$. Suppose $x = (x_i)_{i\in\mathbb{Z}_{\geq 0}}$ and $y = (y_i)_{i\in\mathbb{Z}_{\geq 0}}$ are non-decreasing sequences of non-negative integers. Then the following two items are equivalent,

(i) x and $y_{>0}$ are p-complementary and, for all n, $y_n - x_n = mn$;
(ii) for all n, $x_n = a_n^{m,p}$ and $y_n = b_n^{m,p}$.

Proof. By Lemma 3.3 it is clear that (ii) implies (i). Hence, it suffices to prove the other direction.

It is given that $x_0 = y_0 = a_0 = b_0 = 0$. Since x is non-decreasing the condition $y_n - x_n = mn$ implies that y is increasing. Suppose that Lemma 3.3 (iv) holds for a fixed $n \geq 0$, but with a exchanged for x and b exchanged for y. Then, since x and $y_{>0}$ are p-complementary and $y_{n+1} > x_{n+1}$, we must have that $x_{n+1} - x_n = 0$ if

$$\#\{i \mid x_i = x_n \text{ or } y_{i+1} = x_n, 0 \leq i \leq n\} < p,$$

and $x_{n+1} - x_n = 1$ if

$$\#\{i \mid x_i = x_n \text{ or } y_{i+1} = x_n, 0 \leq i \leq n\} = p.$$

But, by Lemma 3.3, this also holds for the sequence (a_i). In conclusion, $y_{n+1} = x_{n+1} + m(n+1) = a_{n+1} + m(n+1) = b_{n+1}$ gives the result. □

5. Recurrence results

In Proposition 5.2 in this section, we generalize the MEX-algorithm in (5). Each game family in Definition 4 has motivated the study of a 'corresponding' MEX-algorithm. The numberings in Definition 4 and Proposition 5.2 respectively are in accordance with this correspondence.

But first we explain why the sequences a and b satisfy the 'p-complementary equation' in (6).

Theorem 5.1. Fix $m, p \in \mathbb{Z}_{>0}$ and let a and b be as in (2) and (3). For each $n \in \mathbb{Z}_{\geq 0}$, define

$$\varphi_n = \varphi_n(m, p) := \frac{a_n + (mp-1)b_n}{m}.$$

Then, for each $n \in \mathbb{Z}_{>0}$, φ_n is the greatest integer such that

$$b_n - 1 = a_{\varphi_n}. \tag{8}$$

Proof. Notice that, for all n,

$$\begin{aligned}\varphi_n &= \frac{a_n + (mp-1)b_n}{m} \\ &= \frac{mpb_n - mn}{m} \\ &= pb_n - n, \end{aligned} \tag{9}$$

so that

$$
\begin{aligned}
\varphi_{n+1} - \varphi_n &= pb_{n+1} - (n+1) - (pb_n - n) \\
&= p(b_{n+1} - b_n) - 1. \qquad (10)
\end{aligned}
$$

The proof is by induction. For the base case, notice that $b_1 = m$, $a_1 = 0$ and $\varphi_1 = (mp-1)$. The only representative from b in the interval $[0, p-1]$ is $b_0 = 0$ (which we by definition do not count). Hence, by $a_0 = 0$ and p-complementarity, we get that

$$a_{\varphi_1} = a_{mp-1} = m - 1 = b_1 - 1$$

and

$$a_{\varphi_1+1} = a_{mp} = m = b_1.$$

Suppose that (8) holds for all $i \le n$. Then we need to show that $b_{n+1} - 1 = a_{\varphi_{n+1}}$ and $b_{n+1} = a_{\varphi_{n+1}+1}$.

If $a_{\varphi_{n+1}} - a_{\varphi_n} = b_{n+1} - b_n$, by $b_n - 1 = a_{\varphi_n}$ and $b_n = a_{\varphi_n+1}$, we are done, so assume that either

(A) $a_{\varphi_{n+1}} - a_{\varphi_n} < b_{n+1} - b_n$, or
(B) $a_{\varphi_{n+1}} - a_{\varphi_n} > b_{n+1} - b_n$.

Again, by p-complementarity, the total number of elements from a and b in the interval

$$
\begin{aligned}
I_n &:= (a_{\varphi_n}, a_{\varphi_{n+1}}] \\
&= (a_{\varphi_n}, a_{\varphi_n + p(b_{n+1}-b_n)-1)}]
\end{aligned}
$$

is $R_n := p(a_{\varphi_{n+1}} - a_{\varphi_n})$, and where the equality is by (10). By assumption, $a_{\varphi_n+1} \in I_n$ so that we have at least $p(b_{n+1} - b_n) - 1$ representatives from a in I_n. But also $b_n = a_{\varphi_n} + 1 \in I_n$ so that altogether we have at least $p(b_{n+1} - b_n)$ representatives in I_n. Hence

$$
\begin{aligned}
p(b_{n+1} - b_n) &\le R_n \\
&= p(a_{\varphi_{n+1}} - a_{\varphi_n})
\end{aligned}
$$

which rules out case (A).

Notice that case (B) implies that b_{n+1} lies in I_n so that $a_{\varphi_n+1} = b_n < b_{n+1} \le a_{\varphi_{n+1}}$. Since both b_n and b_{n+1} lie in I_n, the total number of representatives in I_n is

$$
\begin{aligned}
p(a_{\varphi_{n+1}} - a_{\varphi_n}) &\le 2 + \varphi_{n+1} - \varphi_n \\
&= p(b_{n+1} - b_n) + 1. \qquad (11)
\end{aligned}
$$

In case $p > 1$, since a and b are integer sequences, we are done, so suppose $p = 1$. Then, in fact, by complementarity, we must have $a_{\varphi_{n+1}} < b_n < b_{n+1} < a_{\varphi_{n+1}}$, contradicting (11). □

Remark 4. For arbitrary $m > 0$ and $p = 1$ it is well-known that a and b solve $x_{y_n} = x_n + y_n$. This complementary equation is studied in for example [Con59, FK94, Ki07]. However, we have not been able to find any references for the complementary equation $y_n - 1 = x_{y_n - n}$. By (9), for the cases $p = 1$, this equation is also resolved by a and b.

A *multiset* (or a sequence) X may be represented as (another) sequence of non-negative integers $\xi = (\xi^i)_{i\in\mathbb{Z}_{\geq 0}}$, where, for each $i \in \mathbb{Z}_{\geq 0}$, $\xi^i = \xi^i(X)$ counts the number of occurrences of i in X. For a positive integer p, let $\operatorname{mex}^p \xi$ denote the least non-negative integer $i \in X$ such that $\xi^i < p$.

Proposition 5.2. Let $m, p \in \mathbb{Z}_{>0}$. Then the definitions of the sequences x and y in (i), (ii) and (iii) are equivalent. In fact, for each $n \in \mathbb{Z}_{\geq 0}$, we have that $x_n = a_n^{m,p}$ and $y_n = b_n^{m,p}$.

(i) For $n \geq 0$,

$$x_n = \operatorname{mex}^p \xi_n,$$

where ξ_n is the multiset, where for each $i \in \mathbb{Z}_{\geq 0}$,

$$\xi_n^i = \#\{j \mid i = x_j \text{ or } i = y_j, 0 \leq j < n\},$$
$$y_n = x_n + mn.$$

(ii) For $n \geq 0$,

$$x_n = \operatorname{mex}\{\nu_i^n, \mu_i^n \mid 0 \leq i < n\}, \text{ where}$$
$$\nu_i^n = x_i \text{ if } n \equiv i \pmod p, \text{ else } \nu_i^n = \infty,$$
$$\mu_i^n = y_i \text{ if } n \equiv -i \pmod p, \text{ else } \mu_i^n = \infty,$$
$$y_n = x_n + mn.$$

(iii) For $n \geq 0$,

$$x_{pn} = \operatorname{mex}\{x_{pi}, y_{pi} \mid 0 \leq i < n\},$$
$$y_{pn} = x_{pn} + mpn,$$

and for each integer $0 < l < p$,

$$x_{pn+l} = \operatorname{mex}\{x_{pi+l}, y_{p(i+1)-l} \mid 0 \leq i < n\},$$
$$y_{pn+l} = x_{pn+l} + m(pn + l).$$

Proof. For $p = 1$ it is a straightforward task to check that each recurrence is equivalent to (5). Hence, let $p > 1$. Observe that in (i), by definition, x and y are non-decreasing, p-complementary and, for all n,

$$y_n = x_n + mn. \tag{12}$$

Hence, for this case, Theorem 4.1 gives the result.

Let us now study the definitions of x and y in (ii). For $z \in \mathbb{Z}$, let $\overline{z}$ denote the congruence class of z modulo p. Here, it is not immediately clear that the sequences are non-decreasing. Neither is it obvious that they are p-complementary. But, at least we have that, for each $n \in \mathbb{Z}_{\geq 0}$, (12) holds.

Hence, if (ii) fails (by $a_0 = x_0$) there has to exist a least index $n' \in \mathbb{Z}_{>0}$ such that $a_{n'} \neq x_{n'}$. But notice that $0 \leq n < p$ implies $\nu_i^n = \mu_i^n = \infty$, for all $0 \leq i < n$, which in its turn implies $a_n = x_n = 0$. This gives $n' \geq p$. We have two cases to consider:

(a) $r := x_{n'} < a_{n'}$: By Theorem 5.1 there are two cases to consider.
Case 1: There is an $i \geq 0$ such that $\varphi(i) + p - 1 < n'$ and

$$y_i = x_{\varphi(i)+1} = x_{\varphi(i)+2} = \ldots = x_{\varphi(i)+p-1} = r.$$

But then, by

$$\{\ \overline{-i},\ \overline{-i+1},\ \dots,\ \overline{-i+p-1}\ \} = \{\ \overline{0},\ \overline{1},\ \dots,\ \overline{p-1}\ \} \tag{13}$$

and

$$\varphi_n = pb_n - n \equiv -n \pmod{p}, \tag{14}$$

there is a $j \in \{i, \varphi(i)+1, \dots, \varphi(i)+p-1\}$ such that either $n' \equiv j \pmod{p}$ and $j \in \{\varphi(i)+1, \dots \varphi(i)+p-1\}$ which implies $\nu_j^{n'} = r$, or $n' \equiv -j \pmod{p}$ and $j = i$ which implies $\mu_j^{n'} = r$. In either case the choice of $x_{n'} = r$ contradicts the definition of mex.

Case 2: There is an $i \geq 0$ such that $i + p - 1 < n'$ and

$$r = x_i = x_{i+1} = x_{i+2} = \dots = x_{i+p-1}.$$

This case is similar but simpler, since for this case we rather use that

$$\{\ \overline{i},\ \overline{i+1},\ \dots,\ \overline{i+p-1}\ \} = \{\ \overline{0},\ \overline{1},\ \dots,\ \overline{p-1}\ \} \tag{15}$$

(b) $r := a_{n'} < x_{n'}$: Then our mex-algorithm has refused r as the choice for $x_{n'}$. But then there must be an index $0 \leq j < n'$ such that either $\nu_j^{n'} = r$ or $\mu_j^{n'} = r$. Hence, we get to consider two cases.

Case 1: $\overline{j}\ = \overline{n'}$ and $r = x_j$. On the one hand, there is a $k \in \mathbb{Z}_{>0}$ such that $kp + j = n'$ On the other hand, there is a greatest $k' \in \mathbb{Z}_{>0}$ such that $a_{n'-k'} = a_{n'-k'+1} = \dots = a_{n'}$ and by p-complementarity $0 \leq k' < p$. But then, since $n'-k' > n'-kp = j$, we get $a_j < r = x_j$, which contradicts the minimality of n'.

Case 2: $\overline{-j}\ = \overline{n'}$ and $r = y_j$. Then, by Theorem 5.1, $\varphi_j + 1$ is the least index such that $a_{\varphi_j+1} = a_{n'}$. Then, by minimality of n', $a_j = x_j$ gives $b_j = y_j$ so that $a_{n'} = b_j$. For this case, p-complementarity gives $n' - (\varphi_j + 1) + 1 \leq p - 1$. Then $0 < k' := n' - \varphi_j < p$ and so

$$\overline{-j + k'} = \overline{\varphi(j) + k'} = \overline{n'} = \overline{-j},$$

which is nonsense.

For case (iii), suppose that there is a least index $n' \geq p$ such that $a_{n'} \neq x_{n'}$. (The case $n' < p$ may be ruled out in analogy with (ii).) Then, there exist unique integers, $0 < t$ and $0 \leq l < p$, such that $tp + l = n'$.

Suppose that $r := a_{n'} < x_{n'}$. Then, since the mex-algorithm did not choose $x_{n'} = r$, there must be an index $0 \leq t' < t$ such that either $x_{t'p+l} = r$ or $y_{(t'+1)p-l} = r$. But then, by assumption, either $a_{t'p+l} = x_{t'p+l} = a_{n'}$ or $b_{(t'+1)p-l} = y_{(t'+1)p-l} = a_{n'}$. But, by Proposition 3.2 a and b are complementary, so either case is ridiculous.

Hence, assume $r := a_{n'} > x_{n'}$. Then again, by Proposition 3.2, there is an index $0 \leq t' < t$ such that either $a_{t'p+l} = x_{n'}$ or $b_{(t'+1)p-l} = x_{n'}$. But, again, by minimality of n', this contradicts the mex-algorithm's choice of $x_{n'} < a_{n'}$. □

6. The games final section

The proof of Theorem 2.1 is based on standard impartial games arguments—with occasional references to facts already established in for example Proposition 5.2. Notice that the blocking variation in (i) is 'simpler' (and more elegant?) than the other, namely it depends only on results from Section 3.

Proof of Theorem 2.1. For $p = 1$, the games have identical rules. This case has been established in [Fr82]. The case $m = 1$ has been studied in [Co59] for games of form (ii). (and implicitly for $1{\times}p\mathrm{WN}_l$).

For the rest of the proof assume that $p > 1$. For each game we need to prove that, if (x, y)

(A) is of the form $\{a_i, b_i\}$, then none of its options is;
(B) is not of the form $\{a_i, b_i\}$, then it has an option of this form.

(We will need a slightly different notation for Case (iii) below.) By symmetry, we may assume that $0 \le x \le y$. Clearly, any final position satisfies (A) but not (B).

Game (i): We need to prove that $\mathcal{P}(m\mathrm{WN}^p) = \{\{a_i, b_i\} \mid i \in \mathbb{Z}_{\ge 0}\}$. Suppose $(x, y) = (a_i, b_i)$ for some $i \in \mathbb{Z}_{\ge 0}$. By Lemma 3.3 (i) and (ii), a and $b_>$ are p-complementary and $b_i - b_j \ge m$ for all $j < i$. Then any roob-type option of the form $\{a_j, b_j\}$ may be blocked off, unless perhaps $a_j < a_i$ and $b_j = b_i$ for some $j < i$. But this is ridiculous since b is strictly increasing. By Lemma 3.3 (ii) we get that, for $j < i$, $b_i - a_i \pm (b_j - a_j) \ge m$. Then an m-bishop cannot move $(x, y) \to \{a_j, b_j\}$, This proves (A).

For (B), since $p \ge 2$, we may assume $x = a_i$, for some i, but $y \ne b_i$. Then, by Lemma 3.3 (iv): (*) There exists a $j < i$ such that an m-bishop can move $(x, y) \to (a_j, b_j)$ unless $y - x - (b_j - a_j) \ge m$ for all j such that $a_j \le x$. Then, for all j such that $a_j = x$, we have that $y \ge x + m(j + 1) > b_j$. But then, by Lemma 3.3 (i), there are p options of (x, y) of the form $\{a_i, bi\}$. By the rule of game, they cannot all be blocked off.

Game (iia): We are going to prove that $\mathcal{P}(m\mathrm{WN}^{(p)}) = \{\{a_i, b_i\} \mid i \in \mathbb{Z}_{\ge 0}\}$ if and only if $\gcd(m, p) = 1$. Let us first explain the 'only if' direction. Denote with $\gamma = \gcd(m, p)$, $p' = \frac{p}{\gamma}$ and $m' = \frac{m}{\gamma}$. Then, clearly, the positions of the form $(0, mi)$, where $0 \le i < p'$, are P-positions of $m\mathrm{WN}^{(p)}$. Now, $(0, mp')$ is an N-position because $m'p = mp'$ implies that $(0, mp') \to (0, 0)$ is an option. But, by definition, $b_{p'} = mp'$ if and only if $p' < p$ if and only if $\gamma > 1$. Hence $\gcd(m, p) = 1$ is a necessary requirement.

For this game, the options of the m-bishop are identical to those in (i). Hence, let us analyze the p-rook.

For (A), suppose that $(x, y) = (a_i, b_i)$ for some $i \in \mathbb{Z}_{\ge 0}$ but that, for a contradiction, that a p-rook can move to $\{a_j, b_j\}$. Then, since b is strictly increasing, there is a $0 \le j < i$, such that either $b_i \equiv b_j \pmod{p}$ and $a_i = a_j$, or $b_i \equiv a_j \pmod{p}$ and $a_i = b_j$. But then, for the first case (using the same notation as in Section 5), since

$$\overline{mj} = \overline{b_j - a_j} = \overline{b_i - a_i} = \overline{mi}$$

and $\gcd(m, p) = 1$ we must have $\overline{j} = \overline{i}$. This is ridiculous, since by p-complementarity and a non-decreasing we have $0 < i - j < p$. For the second case, by Theorem 5.1, we have that

$$\overline{-mj} = \overline{a_j - b_j} = \overline{b_i - a_i} = \overline{mi} = \overline{m(\varphi(j) + t)} = \overline{m(-j + t)},$$

for some $t \in \{1, \ldots, p-1\}$. This implies $\overline{0} = \overline{mt}$ but then again $\gcd(m, p) = 1$ gives a contradiction.

For (B), we follow the ideas in the second part of Case (i) up until (*). Then, for this game, we rather need to show that there is a $j < i$ such that $y \equiv b_j \pmod{p}$ and $a_j = x$ or $y \equiv a_j \pmod{p}$ and $b_j = x$. But this follows directly from the proof of Proposition 5.2 (ii)(a).

Game (iib): We are now going to show that $\mathcal{P}(m\mathrm{WN}^{(m,mp)}) = \{\{a_i, b_i\} \mid i \in \mathbb{Z}_{\geq 0}\}$. For (A), suppose $(x, y) = (a_i, b_i)$ for some $i \in \mathbb{Z}_{\geq 0}$ but that there is a $j < i$ such that the (m, mp)-rook can move to $\{a_j, b_j\}$. Then, we have two cases:

Case 1: $b_i \equiv b_j - r \pmod{mp}$ and $a_i = a_j$, for some $r \in \{0, 1, \ldots, m-1\}$. Then $b_i - a_i \equiv b_j - a_j - r \pmod{mp}$ so that $mi \equiv mj - r \pmod{mp}$ and so $m(i - j) \equiv -r \pmod{mp}$. But this forces $r = 0$ and $i - j \equiv 0 \pmod{p}$ which is impossible since Lemma 3.3 (i) and (iv) imply $i - j \in \{1, 2, \ldots, p-1\}$.

Case 2: $b_i \equiv a_j - r \pmod{mp}$ and $a_i = b_j$, for some $r \in \{0, 1, \ldots, m-1\}$. Then $b_i - a_i \equiv a_j - b_j - r \pmod{mp}$ so that $mi \equiv -mj - r \pmod{mp}$ and so $m(i + j) \equiv -r \pmod{mp}$. By Theorem 5.1 we have that $i = \varphi(j) + s$ for some $s \in \{1, 2, \ldots, p-1\}$. Further, by (14), we have $\varphi(j) \equiv -j \pmod{p}$, so that $m(\varphi(j) + s + j) = ms \equiv -r \pmod{mp}$. Once again we have reached a contradiction.

For (B), in analogy with (*), it suffices to study the (m, mp)-rook's options where y is such that $y - x - (b_j - a_j) \geq m$ for all j such that $a_j \leq x = a_i$. Hence, we need to show that there are a j and an $r \in \{0, 1, \ldots m-1\}$ such that

$$y \equiv b_j - r \pmod{mp} \quad \text{and} \quad a_j = x,$$

or

$$y \equiv a_j - r \pmod{mp} \quad \text{and} \quad b_j = x.$$

Clearly, we may choose r such that $y - x + r \equiv 0 \pmod{m}$. Then, for all j, we get $ms := y - x + r \equiv \pm(b_j - a_j) \pmod{m}$. Hence, it suffices to find a specific j such that

$$j = \frac{b_j - a_j}{m} \equiv s \pmod{p} \quad \text{and} \quad a_j = x,$$

or

$$-j = \frac{a_j - b_j}{m} \equiv s \pmod{p} \quad \text{and} \quad b_j = x.$$

But then, by (13) or (15), we are done.

Game (iiia): We are now going to show that $\mathcal{P}(m{\times}p\mathrm{WN}_l) = \{(a_{ip+l}, b_{ip+l}) \mid i \in \mathbb{Z}_{\geq 0}\} \cup \{(b_{ip-l}, a_{ip-l}) \mid i \in \mathbb{Z}_{>0}\}$. We may assume that $l > 0$. We have already seen that $(a'_i) := (a_{pi+l})_{i \geq 0}$ and $(b'_i) := (b_{p(i+1)-l})_{i \geq 0}$ are complementary. Our proof will be a straightforward extension of those in [Fr82] (which deals with the case $l = 0$) and [Co59] (which implicitly deals with the case $m = 0$). Observe that $a'_0 = a_l = 0$ and $b'_0 = b_{p-l} = m(p - l)$.

For (A), let $(x, y) = (a'_i, b'_i)$. In case $i = 0$ (by Definition 4 (iiia)), the Queen has no options at all, so assume $i > 0$. Proposition 5.2 (iii) gives that $b'_i - a'_i \pm (b'_j - a'_j) \geq mp$ for all $0 \leq j < i$. Then the mp-bishop cannot move $(x, y) \to (a'_j, b'_j)$ for any $0 \leq j < i$. Since a' and b' are complementary there is no rook-type option $(a'_i, b'_i) \to \{a'_j, b'_j\}$.

For (B), we adjust the statement (*) accordingly: Suppose $x = a'_i$. By Proposition 5.2 (iii): If the mp-bishop cannot move to (a'_j, b'_j) for any $j < i$ we get that either $i = 0$ or $y - x - (b'_j - a'_j) \geq mp$ for all $j < i$. If $i = 0$ there is a rook-type option to (a'_0, b'_0) (we may assume here that $y > b'_0$), so suppose $i > 0$. But then, by Proposition 5.2 (iii), we get $y \geq b'_j + mp + x - a'_j \geq b'_i + a'_i - a'_j > b'_i$. Hence, for this case, the rook-type move $(x, y) \to (a'_i, b'_i)$ suffices. Suppose on the other hand that $x = b'_i$ with $i \geq 0$. Then, since $y \geq x = b'_i > a'_i$, the desired rook-type move is $(x, y) \to (b'_i, a'_i)$.

Game (iiib): It only remains to demonstrate that $\mathcal{P}(m{\times}p\text{WN}) = \{\{a_i, b_i\} \mid i \in \mathbb{Z}_{\geq 0}\}$. Suppose that the starting position is (a_i, b_i). Then $i = pj + l'$ for some (unique) pair $j \in \mathbb{Z}_{\geq 0}$ and $0 \leq l' < p$. The second player should choose $l = l'$. If, on the other hand, the starting position is (b_i, a_i). Then $i = pj - l'$ for some (unique) pair $j \in \mathbb{Z}_{>0}$ and $0 < l' \leq p$. The second player should choose $l = p - l'$. In either case, by (iiia), there is no option of the form (a'_i, b'_i).

If the starting position (x, y) is not of the form $\{a_i, b_i\}$, again, by (iiia), for any choice of $0 \leq l < p$, there is a move $(x, y) \to \{a'_i, b'_i\}$ for some $i \geq 0$.

□

7. Questions

Can one find a polynomial time solution of $m\text{WN}^{(l,p)}$ for some integers $l \geq 0$, $m > 0$ and $p > 0$ whenever

- $\gcd(m, p) \neq 1$ and $l = 0$, or
- $0 < l \neq m$ or $m \nmid p$?

If this turns out to be complicated, can one at least say something about its asymptotic behavior?

Denote the solution of $m\text{WN}^{(l,p)}$ with $\{\{c_i^{(l,m,p)}, d_i^{(l,m,p)}\}\}_{i \in \mathbb{Z}_{\geq 0}}$. Let us finish off with two tables of the initial P-positions of such games.

From these tables one may conclude that: The infinite arithmetic progressions of the sequences

$$(b_i^{m,p} - a_i^{m,p})_{i \in \mathbb{Z}_{\geq 0}} = (mi)_{i \in \mathbb{Z}_{\geq 0}}$$

(see also Table 1 and 2) are not in general seen among the sequences

$$(d_i^{(l,m,p)} - c_i^{(l,m,p)})_{i \in \mathbb{Z}_{\geq 0}}.$$

$d_n^{(0,2,2)}$	0	3	6	9	12	15	19	22	25	28	31	34	37	40	43	46	49
$c_n^{(0,2,2)}$	0	0	1	1	2	2	3	4	4	5	5	6	7	7	8	8	9
$d_n - c_n$	0	3	5	8	10	13	16	18	21	23	26	28	30	33	35	38	40
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

TABLE 3. The first few P-positions of $2\mathrm{WN}^2$ together with the respective differences of their coordinates.

$d_n^{(1,2,3)}$	0	2	5	7	11	14	16	19	21	26	29	31	36	39	41	44	46
$c_n^{(1,2,3)}$	0	0	1	1	2	3	3	4	4	5	6	6	7	8	8	9	9
$d_n - c_n$	0	2	4	6	9	11	13	15	17	21	23	25	29	31	33	35	37
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

TABLE 4. The first few P-positions of $2\mathrm{WN}^{(1,3)}$. Notice that (as in Table 3) the successive differences of their coordinates are not in arithmetic progression.

We believe that the latter sequence is an arithmetic progression if and only if none of the items in our above question is satisfied. We also believe that, for arbitrary constants, $(c_i^{(l,m,p)})_{i\in\mathbb{Z}_{\geq 0}}$ and $(d_i^{(l,m,p)})_{i\in\mathbb{Z}_{>0}}$ are p-complementary. But the solution of these questions are left for some future work.

Remark 5. We may also define generalizations of $m\mathrm{WN}^p$ and $m \times p\mathrm{WN}_l$:

Fix $l \in \mathbb{Z}_{>0}$. Let $m\mathrm{WN}_l^p$ be as $m\mathrm{WN}^p$ but where the player may only block off *l-roob-type options* (recall, non-l-bishop options). Otherwise, the Queen moves as the m-bishop or the rook. Then $m\mathrm{WN}_m^p = m\mathrm{WN}^p$. On the other hand $m\mathrm{WN}_1^p$ is the blocking variation of m-Wythoff Nim where the previous player may block off *any* $p-1$ rook-type options.

Let $u, v \in \mathbb{Z}_{>0}$ and let $m \times p\mathrm{WN}_{u,v}$ be as $m \times p\mathrm{WN}_l$, but the removed (lower left) rectangle has base u and height v. Then for this game the final positions are $(u, 0)$ and $(0, v)$. If $l > 0$, $u = ml$ and $v = m(p-l)$ we get $m \times p\mathrm{WN}_{lm,m(p-l)} = m \times p\mathrm{WN}_l$. Some of these games are identical to misère versions of Wythoff Nim, see [Fr84].

One may ask questions in analogy to the above for these variations. For example, we have found a minimal exclusive algorithm satisfying $\mathcal{P}(m\mathrm{WN}_1^p)$ which is related to a polynomial time construction in [Fr98]. Is there an analog polynomial time construction for $\mathcal{P}(m\mathrm{WN}_1^p)$? Another question is if any of these further generalized games coincide via identical set of P-positions?

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References

[ANW07] M. H. Albert, R. J. Nowakowski, D. Wolfe *Lessons in Play: In Introduction to Combinatorial Game Theory.* A K Peters Ltd.(2007).

[AMM] Solution II, Problem 11365, *Amer. Math. Monthly*, (April 2010), p. 376,

[Be26] S. Beatty, Problem 3173, *Amer. Math. Monthly*, **33** (1926) 159.

[BCG82] E. R. Berlekamp, J. H. Conway, R.K. Guy, *Winning ways*, **1-2** Academic Press, London (1982). Second edition, **1-4**. A. K. Peters, Wellesley/MA (2001/03/03/04).

[BB93] J. M. Borwein and P. B. Borwein, On the generating function of the integer part: $[\alpha n + \gamma]$, *J. Number Theory* 43 (1993), pp. 293-318.

[BoFr73] I. Borosh, A.S. Fraenkel, A Generalization of Wythoff's Game, *Jour. of Comb. Theory* (A) **15** (1973) 175-191.

[BF84] M. Boshernitzan and A. S. Fraenkel, A linear algorithm for nonhomogeneous spectra of numbers, *J. Algorithms*, **5**, no. 2, pp. 187-198, 1984.

[Bo02] C.L. Bouton, Nim, a game with a complete mathematical theory, *The Annals of Math. Princeton* (2) **3** (1902), 35-39.

[O'B02] K. O'Bryant, A Generating Function Technique for Beatty Sequences and Other Step Sequences, *J. Number Theory* 94, 299–319 (2002).

[O'B03] K. O'Bryant, Fraenkel's Partition and Brown's Decomposition *Integers*, **3** (2003), A11, 17 pp.

[Co59] I.G. Connell, A generalization of Wythoff's game *Can. Math. Bull.* **2** no. 3 (1959), 181-190.

[Con59] I.G. Connell, Some properties of Beatty sequences I *Can. Math. Bull.* **2** no. 3 (1959), 190-197.

[C76] J. H. Conway: *On numbers and games*, Academic Press, London (1976). Second edition, A. K. Peters, Wellesley/MA (2001).

[DG08] E.Duchêne, S. Gravier, Geometrical Extensions of Wythoff's Game, to appear in *Discrete Math* (2008).

[Fr69] A.S. Fraenkel, The bracket function and complementary sets of integers, *Canad. J. Math.* **21** (1969), 6-27.

[Fr73] A.S. Fraenkel, Complementing and exactly covering sequences, J. Comb. Theory (Ser A), **14** (1973) 8-20.

[Fr82] A.S. Fraenkel, How to beat your Wythoff games' opponent on three fronts, *Amer. Math. Monthly* **89** (1982) 353-361.

[Fr84] A.S. Fraenkel, *Wythoff games, continued fractions, cedar trees and Fibonacci searches*, Theoret. Comput. Sci. 29 (1984) 49-73.

[Fr98] A.S. Fraenkel, Heap Games, Numeration Systems and Sequences. *Ann. of Comb.*, **2** (1998) 197-210.

[Fr04] A.S. Fraenkel, Complexity, appeal and challenges of combinatorial games. *Theoret. Comp. Sci.*, **313** (2004) 393-415.

[FP09] A.S. Fraenkel, Udi Peled, Harnessing the Unwieldy MEX Function, preprint, http://www.wisdom.weizmann.ac.il/ fraenkel/Papers/ Harnessing.The.Unwieldy.MEX.Function_2.pdf.

[GS04] H. Gavel and P. Strimling, Nim with a Modular Muller Twist, *Integers: Electr. Jour. Comb. Numb. Theo.* **4** (2004).

[Ha] U. Hadad, Msc Thesis, Polynomializing Seemingly Hard Sequences Using Surrogate Sequences, *Fac. of Math. Weiz. In. of Sci.*, (2008).

[HL06] P. Hegarty and U. Larsson, Permutations of the natural numbers with prescribed difference multisets, *Integers* **6** (2006), Paper A3, 25pp.

[HR] A. Holshouser and H. Reiter, Three Pile Nim with Move Blocking, http://citeseer.ist.psu.edu/470020.html.

[HO27] A. Ostrowski and J. Hyslop, Solution to Problem 3177, *Amer. Math. Monthly*, **34** (1927), 159-160.

[FK94] A.S. Fraenkel and C. Kimberling, Generalised Wythoff arrays, shuffles and interspersions, *Discrete Math.* **126** (1994), 137-149.

[Ki95] C. Kimberling, Stolarsky interspersions, *Ars Combinatoria* **39** (1995), 129-138.

[Ki07] C. Kimberling, Complementary equations, *J. Integer Sequences* **10** (2007), Article 07.1.4.

[Ki08] C. Kimberling, Complementary equations and Wythoff sequences, *J. Integer Sequences* **11** (2008), Article 08.3.3.

[La09] U. Larsson, 2-pile Nim with a Restricted Number of Move-size Imitations, *Integers* **9** (2009), Paper G4, 671-690.

[Ra94] J. W. Rayleigh. The Theory of Sound, *Macmillan, London*, (1894) p. 122-123.

[Sk57] Th. Skolem, Über einige Eigenschaften der Zahlenmengen $[\alpha n + \beta]$ bei irrationalem α mit einleitenden Bemerkungen über eine kombinatorishe Probleme, *Norske Vid. Selsk. Forh., Trondheim* **30** (1957), 42-49.

[SS02] F. Smith and P. Stănică, Comply/Constrain Games or Games with a Muller Twist, *Integers*, **2**, (2002).

[Wy07] W.A. Wythoff, A modification of the game of Nim, *Nieuw Arch. Wisk.* **7** (1907) 199-202.

Appendix

The following discussion, provided by Aviezri Fraenkel, provides a complementary analysis of 'p-complementarity'/'p-fold complementarity' and homogeneous Beatty sequences:

Definition 1. Let $p \in \mathbb{Z}_{>0}$. The multisets S, T of positive integers are 1-*upper* p*-fold complementary*, for short: p*-fold complementary*, if $S \cup T = p \times \mathbb{Z}_{>0}$.

If the multisets S, T satisfy Definition 1 and have irrational densities α^{-1}, β^{-1}, say $\alpha \leq \beta$, then a necessary condition for p-fold complementarity is $\alpha^{-1} + \beta^{-1} = p$. Thus $a := \beta - \alpha > 0$. Then $\alpha = (2 - ap + \sqrt{a^2p^2 + 4})/2p$, so $p^{-1} < \alpha < 2p^{-1}$. Then $1/\beta = p - 1/\alpha$, so $\beta > 2/p$.

Let $M = \lfloor 1/\alpha \rfloor + 1$, $N = \lfloor 1/\beta \rfloor + 1$. Notice that $\alpha(M-1) < 1 < \alpha M$, $\beta(N-1) < 1 < \beta N$. From now on we let $S = \{\lfloor n\alpha \rfloor\}_{n \geq M}$, $T = \{\lfloor n\beta \rfloor\}_{n \geq N}$.

Theorem 1. The multisets S, T are p-fold complementary.

Proof. For any $k \in \mathbb{Z}_{>0}$, since α is irrational, the number of terms less than k in $S \cup T$ is

$$\begin{aligned}\lfloor k/\alpha \rfloor - (M-1) + \lfloor k/\beta \rfloor - (N-1) &= \lfloor k/\alpha \rfloor + \lfloor k(p - \alpha^{-1}) \rfloor - M - N + 2 \\ &= kp + \lfloor k/\alpha \rfloor + \lfloor -k/\alpha \rfloor - M - N + 2 \\ &= kp - M - N + 1.\end{aligned}$$

Similarly, $S \cup T$ contains $(k+1)p - M - N + 1$ terms $< k+1$. Hence there are exactly p terms $< k+1$ but not $< k$. They are the terms k with multiplicity p. □

Remarks. (i) $\lfloor 1/\alpha \rfloor = p + \lfloor -1/\beta \rfloor = p - 1 - \lfloor 1/\beta \rfloor = p - N$. Hence $M = \lfloor 1/\alpha \rfloor + 1 = p - N + 1$.

(ii) Clearly $(\lfloor (M-1)\alpha \rfloor, \lfloor (N-1)\beta \rfloor) = (0, 0)$. Since $\alpha < \beta$, we have $N \leq M$. Hence, for all $N \leq n < M$, we have that $(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor) = (0, \lfloor n\beta \rfloor)$ where $\lfloor n\beta \rfloor > 0$. Thus there are precisely $M - N$ couples $(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$ with $\lfloor n\alpha \rfloor = 0$ and $\lfloor n\beta \rfloor > 0$, containing $M - N$ 0s. Thus, for $0 \leq n < M$, there are precisely M couples $(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$ with $\lfloor n\alpha \rfloor = 0$ and $\lfloor n\beta \rfloor \geq 0$, containing, in total, $M + N = p + 1$ 0s.

(iii) The proof is a straightforward generalization to $p \geq 1$ of a proof included in an editorial comment to [AMM] stating: "...The result is so astonishing and yet easily proved that we include a short proof for the reader's pleasure." This is then followed by the above proof for the special case $p = 1$, which is itself a slight simplification of the proof given in [Fr82].

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