THE $\star$-OPERATOR AND INVARIANT SUBTRACTION GAMES

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Abstract. We study 2-player impartial games, so called *invariant subtraction games*, of the type, given a set of allowed moves the players take turn in moving one single piece on a large Chess board towards the position $\mathbf{0}$. Here, invariance means that each allowed move is available inside the whole board. Then we define a new game, $\star$ of the old game, by taking the P-positions, except $\mathbf{0}$, as moves in the new game. One such game is $W^\star$ = (Wythoff Nim)*, where the moves are defined by complementary Beatty sequences with irrational moduli. Here we give a polynomial time algorithm for infinitely many P-positions of $W^\star$. A repeated application of $\star$ turns out to give especially nice properties for a certain subfamily of the invariant subtraction games, the *permutation games*, which we introduce here. We also introduce the family of *ornament games*, whose P-positions define complementary Beatty sequences with rational moduli—hence related to A. S. Fraenkel's 'variant' Rat- and Mouse games—and give closed forms for the moves of such games. We also prove that (k-pile Nim)** = k-pile Nim.

1. Introduction and terminology

This paper is a sequel to [LHF]. We begin by recapitulating some notation and terminology. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A 2-player *impartial* [BCG82] game is a combinatorial game where, independent of whose turn it is, the options are the same. Here (as in [LHF]) we study so called *invariant subtraction games*[1], impartial 'board games', mostly played on the board $\mathcal{B} = \mathbb{N}_0 \times \mathbb{N}_0$ (except in the last section where it is $\mathbb{N}_0^k$, $k \in \mathbb{N}$). Given a set of 'invariant moves', denoted by $\mathcal{M}(G)$, the two players take turn in moving a single piece towards the position $\mathbf{0} = (0, 0)$. The player who moves there wins. A 'move' is represented by an ordered pair of non-negative integers, say $(i, j) \neq \mathbf{0}$. In practice, this ordered pair is subtracted from the piece's current position, say (x, y). The resulting position of this move, provided it is allowed, is

$$(x, y) \ominus (i, j) \succeq \mathbf{0}.$$

Here *invariance* (of the the set of moves) means that each allowed move is playable from any position of $\mathcal{B}$, provided that the piece remains on the board.

Nim [Bo02] is a classical impartial game played on a finite number of piles each containing a finite number of tokens. The players take turn in

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[1]The subtraction games in [BCG82] are special cases of the ones studied here.

removing tokens from precisely one of the piles, at least one token and at most the whole pile. It may be regarded as an invariant subtraction game with, if played on two piles,

$$\mathcal{M}(\text{2-pile Nim}) = \{\{0, x\} \mid x \in \mathbb{N}\}.$$

(We use the 'symmetric notation' $\{x, y\}$ whenever the ordered pairs (x, y) and (y, x) are considered the same.) Another classical example of an invariant subtraction game is Wythoff Nim [Wy07], here denoted by W. The players take turn in moving a Queen of Chess on a large Chess board towards the lower-left corner. In our notation, the moves are

$$\mathcal{M}(\text{W}) = \mathcal{M}(\text{2-pile Nim}) \cup \{(x, x) \mid x \in \mathbb{N}\}.$$

As many impartial games, invariant subtraction games have no draw (cyclic) moves and hence the positions are either P (the previous player wins) or N (the next player wins). Given a game G, the sets of all P-positions and all N-positions is denoted by $\mathcal{P}(G)$ and $\mathcal{N}(G)$ respectively. We denote the set of *terminal* positions by $T(G) \subset \mathcal{P}(G)$. It contains all positions with empty sets of options.

1.1. **Non-zero P-positions as moves.** We may now define the $\star$-operator, introduced in [LHF]. Suppose that G is an invariant subtraction game. Then $G^\star$ is the game defined by

$$\mathcal{M}(G^\star) = \mathcal{P}(G) \setminus \{\mathbf{0}\}.$$

We let G^k denote the resulting game of k recursive applications of $\star$, so that for example $G^0 = G$ and $G^3 = ((G^\star)^\star)^\star$. For the special case of $k = 2$ we prefer to write $G^{\star\star}$. As in [LHF], if $G = G^{\star\star}$ we say that $G^\star$ is the *dual* of G.

1.2. **Games defined via complementary Beatty sequences.** Two sequences of positive integers (a_i) and (b_i) are *complementary* if $\{a_i\} \cup \{b_i\} = \mathbb{N}$ and $\{a_i\} \cap \{b_i\} = \emptyset$. A *Beatty sequence* is a sequence of the form $(\lfloor \alpha n + \gamma \rfloor)$, where $\alpha, \gamma \in \mathbb{R}$ and where n ranges over $\mathbb{N}$. Suppose we have a pair of Beatty sequences, say

$$(\lfloor \alpha n + \delta \rfloor) \text{ and } (\lfloor \beta n + \gamma \rfloor). \tag{1}$$

Necessary and sufficient conditions on their respective moduli and offsets for them to be complementary are given in [Fr69, O'B03].

It is well-known that the set of P-positions of Wythoff Nim may be defined via complementary Beatty sequences with irrational moduli, namely $\mathcal{P}(\text{W}) = \{\{\lfloor \frac{\sqrt{5}-1}{2} n \rfloor, \lfloor \frac{\sqrt{5}+1}{2} n \rfloor\} \mid n \in \mathbb{N}_0\}$. We give a polynomial time algorithm for infinitely many P-positions of the dual $\text{W}^\star$ [LHF, Main Theorem] of Wythoff Nim (which corresponds to infinitely many moves of $\text{W}^{\star\star}$) see also Figure 1. We give a closed formula for the set of moves of the invariant subtraction game, 'the Mouse trap' [LHF]. Here the set of P-positions $\{\{\lfloor \frac{3n}{2} \rfloor, 3n-1\} \mid n \in \mathbb{N}\} \cup \{\mathbf{0}\}$ is defined via complementary Beatty sequences with rational moduli. (Thus, this game has the same P-positions as the 'variant' Mouse game introduced in [Fr08]). We present some more general results on the family of all invariant subtraction games for which the

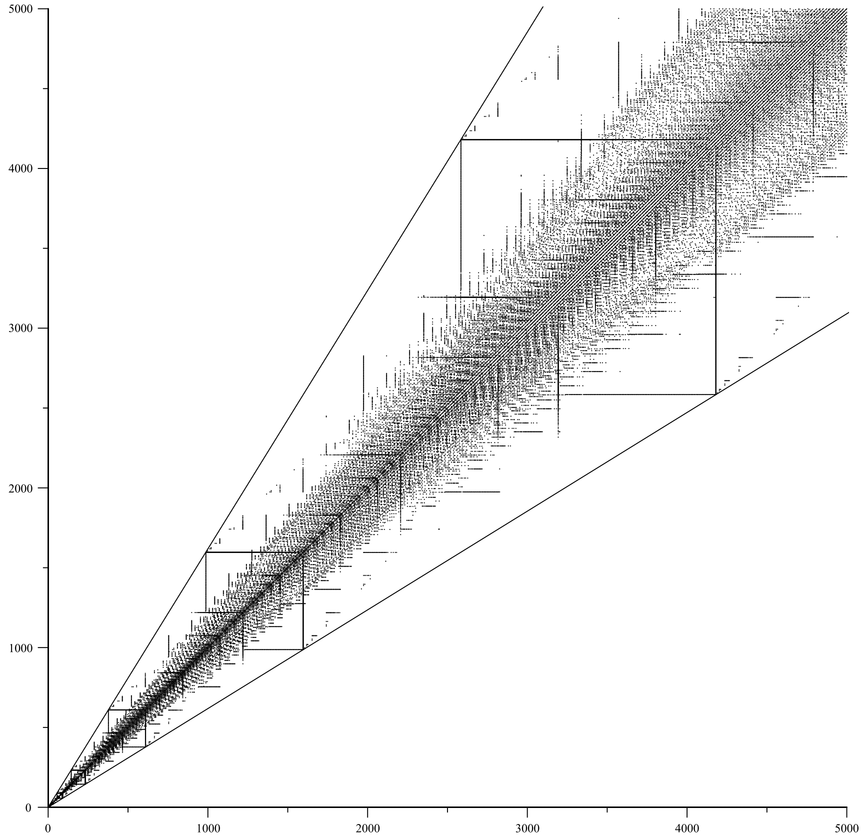

FIGURE 1. The P-positions of (Wythoff Nim)* with coordinates less than 5000 together with the lines through the origin with slopes ϕ and ϕ^{-1} respectively. (Remark: There are no P-positions 'on' these lines.) See also Table 1. In Theorem 2 we prove the existence of infinitely many 'log-periodic' P-positions.

sets of P-positions consists of Beatty sequences with rational moduli, here we introduce the notion of *ornament games* (e.g. Figure 2).

We count the number of such games contained in certain classes and thereby demonstrate that, in total, there are only countably many ornament games. (In contrast, as noted already in [LHF], there are uncountably many invariant subtraction games with sets of P-positions defined via irrational Beatty sequences.) Then, we state a conjecture on invariant subtraction games defined via complementary Beatty sequences saying the the set of P-positions is 'periodic' if and only if the moduli of the respective Beatty sequences is rational.

1.3. **Permutation and involution games.** Let us here introduce the notion of a *permutation* game. This is an invariant subtraction game, where each row and column of $\mathbb{N} \times \mathbb{N}$ contain precisely one move, but, where both row 0 and column 0 are void of moves. We say that a set, say $S \subset \mathcal{B}$, is

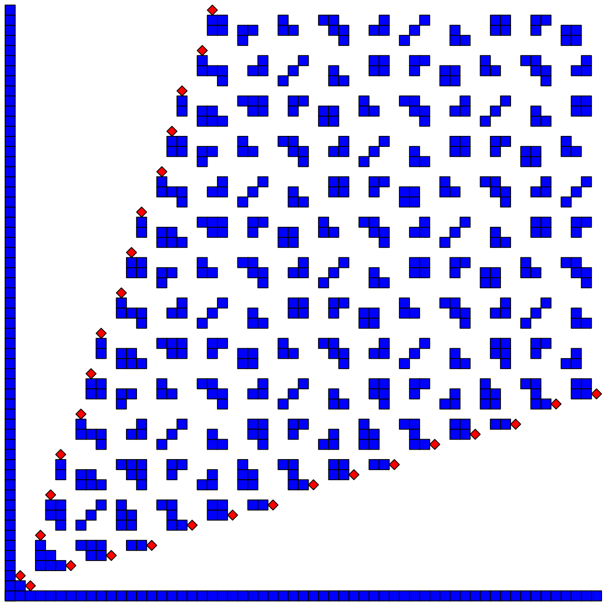

FIGURE 2. For this game, the red diamonds represent moves given by the pairs of complementary Beatty sequences $(\lfloor \frac{4n+1}{3} \rfloor)$ and $(4n-2)$, where n runs over the positive integers. By this we mean that, given a position $(x, y) \in \mathcal{B}$, each legal option is of the form $(x, y) \ominus \{\lfloor \frac{4n+1}{3} \rfloor, 4n-2\} \succeq \mathbf{0}$. The apparent 'periodicity' of the P-positions (which stands in bright contrast to the graph of the P-positions of W* see Figure 1) motivates a 'periodicity'-conjecture on games defined by complementary Beatty sequences (Conjecture 3). See also Figure 3, 4 and 5. By [LHF, Main Theorem], the dual definition is that the blue squares, except $\mathbf{0}$, represent the first few moves of a game where the red diamonds represent the first few non-zero P-positions. Indeed this gives one of the four Class 4 ornament games defined in Section 5.

symmetric if $(x, y) \in S$ if and only if $(y, x) \in S$. An *involution* game is a permutation game where the set of moves is symmetric. Our main results on permutation games are the following. Let G denote a permutation game. Then $G^{\star\star}$ is also, so the permutation games are closed under the operation $\star\star$. In fact, even more is true. The sequence $(G^{2k})_{k\in\mathbb{N}}$ 'converges' and, by the closure property, the resulting game is a permutation game. Similar results hold for involution games.[2]

1.4. **Nim and its dual.** The P-positions of Nim on k piles can be taken as moves in a new game, Nim*. We prove that the non-zero P-positions of Nim* are the moves of Nim. A side effect of this result is a winning strategy of Nim without the mention of 'Nim sum'.

[2]These results were posed as questions on a seminar I gave in the Spring 2010, first at CANT 2010 and then at Dalhousie University. [Pres10]

1.5. **Exposition.** In Section 2 we prove some very basic results for invariant 2-pile subtraction games on complementary sequences of positive integers. In Section 3 we discuss the winning strategy of W*. In Section 4 we study permutation games. In Section 5 we study the invariant subtraction game 'the Mouse trap', including a relative to this game with a so-called 'Muller twist', complementary Beatty sequences with rational moduli and the family of ornament games. In Section 6 we study Nim*.

2. Invariant subtraction games defined by complementary sequences

We begin with a very basic result concerning invariant subtraction games for which the set of moves is defined via complementary sequences of positive integers.

Theorem 1. *Let $a = (a_i)_{i\in\mathbb{N}}$ and $b = (b_i)_{i\in\mathbb{N}}$ denote complementary sequences of positive integers, a increasing, and for all i, $a_i < b_i$. Define G by $\mathcal{M}(G) = \{\{a_i, b_i\} \mid i \in \mathbb{N}\}$. Then,*

- *(i) $(x, y) \in \mathcal{P}(G) \setminus \mathcal{T}(G)$ implies that there is an $i \in \mathbb{N}$ such that $x = a_i$ or $y = a_i$.*
- *(ii) if b is increasing, then $(x, y) \in \mathcal{P}(G) \setminus \mathcal{T}(G)$ implies that there are $i, j \in \mathbb{N}$ such that $x = a_i$ and $y = a_j$.*
- *(iii) if b_i/a_i is bounded by some constant, say $C \in \mathbb{R}$, then $(x, y) \in \mathcal{P}(G) \setminus \mathcal{T}(G)$ (with $x \le y$) implies that $y/x \le C$.*

Proof. By definition of G, all positions of the form $\{0, x\}$, $x \in \mathbb{N}_0$, are terminal and hence P.

Case (i). Suppose $i, j \in \mathbb{N}$ with $i \le j$. Then $(b_i, b_j) \ominus (b_i, a_i)$ is terminal, hence $\{b_i, b_j\}$ is N. The claim follows by complementarity of a and b.

Case (ii). If the claim does not hold, then there is an $i \in \mathbb{N}$ is such that

(a) $y = b_i$, or
(b) $x = b_i$,

But the game is symmetric so it suffices to investigate Case (a). Suppose, in addition, that $x \ge a_i$. Then the option $(x, b_i) \ominus (a_i, b_i)$ is terminal. Otherwise, by complementarity, there is a $j < i$ such that either $a_j = x$ or $b_j = x$. If $b_j = x$, then $b_i > a_j$ so that the option $(b_j, b_i) \ominus (b_j, a_j)$ is terminal. Suppose rather that $x = a_j$. Then, since b is increasing, the option $(a_j, b_i) \ominus (a_j, b_j)$ is legal and hence terminal. By symmetry we may conclude that no position of the form $\{x, b_i\}$ can be P.

Case (iii). Suppose that $y/x > C$. Then, by complementarity, there is an i such that either $b_i = x$ or $a_i = x$. In the first case, $(x, y) \ominus (b_i, a_i)$ is the desired terminal option and in the second case, since $y \ge b_i$, $(x, y) \ominus (a_i, b_i)$ is. We are done. ■

3. A polynomial time algorithm for infinitely many P-positions of W*

Let $\phi := \frac{1+\sqrt{5}}{2}$ denote the golden ratio and, for all $n \in \mathbb{N}_0$, define

$$A_n := \lfloor \phi n \rfloor$$

and

$$B_n := A_n + n.$$

We keep this notation for the rest of this section. Then $\mathcal{P}(\mathrm{W}) = \{\{A_i, B_i\} \mid i \in \mathbb{N}_0\}$ [Wy07]. Thus we have a polynomial time algorithm in $\log n$ for Wythoff Nim's decision problem: Determine whether a given pair of natural numbers represents a P-position.

Let $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}$, $(n \geq 2)$ denote the sequence of Fibonacci numbers. The main result of this section is.

Theorem 2. *For all n, provided both coordinates are positive, the following positions (and its symmetric counterparts) belong to $\mathcal{P}(W^\star)$*

(i) $(F_{2n} - 1, F_{2n} - 1)$,
(ii) $(F_{2n-1}, F_{2n} - 1)$,
(iii) $(F_{2n-1}, F_{2n} - 4)$,
(iv) $(F_{2n-1}, F_{2n} - 9)$,
(v) $(F_{2n-1} + 1, F_{2n} - 1)$,
(vi) $(F_{2n-1} + 3, F_{2n} - 1)$,
(vii) $(F_{2n-1} + 4, F_{2n} - 1)$ *and*
(viii) $(F_{2n-1} + 6, F_{2n} - 1)$.

In this section we make frequent use of the Fibonacci numeration system. Namely, each non-negative integer can be represented as a sum of distinct Fibonacci numbers. Hence we may code any non-negative integer by some binary string

$$\alpha_n \alpha_{n-1} \ldots \alpha_1 := \sum_{i=1}^{n} \alpha_i F_i,$$

for some $n \in \mathbb{N}$ and where, for all i, $\alpha_i \in \{0, 1\}$.

The Zeckendorf numeration system is the unique Fibonacci numeration, where the binary string contains no two consecutive ones.

However, certain properties of a number do not depend on which Fibonacci system of numeration we have used.

Proposition 1. *Let $X \in \mathbb{N}$. Then a Fibonacci coding of X, X_{fib}, ends in an even number of 0s if and only if its Zeckendorf coding, X_{zeck}, does.*

Proof. Search the digits of X_{fib} from left to right. Whenever two consecutive 1s are detected exchange "11" for "100" (where the least 0 has the same position as the previous least 1). Repeat this step until no more "11"s are detected. Then the parity of the number of rightmost 0s in X_{fib} is the same as in the 'output', X_{zeck}. ∎

Lemma 1 ([Fr82])**.** *Let A_n and B_n be defined as above. Then, in Fibonacci coding, A_n ends in an even number of 0s and $B_n = A_n0$.*

Combining this result with Theorem 1 (ii) we obtain the following nice property for the strategy of $W^\star$. This result was first proved by A. S. Fraenkel.

Corollary 1 (A.S.Fraenkel). *Suppose (x, y) represents a P-position of $W^\star$. Then, in Fibonacci coding, both x and y end in an even number of 0s.*

In itself, this result does not reduce the complexity of the decision problem for $W^\star$ to polynomial time[3] in $\log n$. However, it is clear that it characterizes a substantial fraction of the N-positions in polynomial time. In fact, as we will see, this constitutes one of the primary tools for proving polynomial complexity of *certain* P-positions.

The next two results concerns arithmetical properties of numbers of the form $F_{2n} - 1$ and F_{2n-1} respectively. In the 'Fibonacci coding' of a number, we let the symbol 0^t denote a repetition of t consecutive 0s or, for that matter, we let x^t denote a consecutive repetition of t x:s (for example $F_{12} + F_6 + F_4 + F_2 = 100000101010 = 10^5(10)^3$).

Lemma 2. *Let $n \in \mathbb{N}$ and let $X \in \mathbb{N}$ be such that $F_{2n} - X > 0$. In Fibonacci coding, put*

$$\xi := F_{2n} - 1 - X \geq 0.$$

Then

- *ξ ends in an odd number of 0s if X does,*
- *ξ ends in an even number of 0s, namely zero, if X ends in a strictly positive even number of 0s.*

Proof. Recall that,

$$2F_1 = F_2 \text{ and } 2F_n = F_{n+1} + F_{n-2}, \tag{2}$$

for $n \geq 2$. We have that $Y := F_{2n} - 1 = (10)^{n-1}1$. At first suppose that $X = 10^{2t+1}$, $t \in \mathbb{N}$. Then, if $t = 0$, ξ ends in three 0s, otherwise it ends in precisely one 0. Otherwise we must have, in Zeckendorf coding, $X = x010^{2t+1}$, $t \in \mathbb{N}$ for some bit-string $x > 0$. so we need to study an expression of the form $Y - X = (10)^{n-1} - x010^{2t+1}$. The trick we have in mind is probably easiest seen via an example: Put $n = 6$ so that $Y = F_{12} - 1 = 10101010101$ and suppose that $X = x0100000$,which gives $t = 2$. Then, repeated application of (2) give

$$\begin{aligned}\xi &= 10101010101 - x0100000\\ &= 1201010101 - x0100000\\ &= 1112010101 - x0100000\\ &= 1111120101 - x0100000\\ &= z020101\\ &= z100201\\ &= z101010.\end{aligned}$$

[3]For small n (≤ 50000), it does give a considerable improvement in computing capacity. In fact, it seems that then the bounds on the memory (storage of P-positions) sets the limit of computation rather than the processing power.

This number ends in an odd number of 0s irrespective of z, namely precisely one. This trick holds for all n and X which satisfy the conditions of the Lemma, except if X ends in precisely one 0. Hence we need to study this case separately. Here we get

$$\xi = 1^{2(n-1)}2 - x010 = z102,$$

where x and z are bit-strings in Zeckendorf coding both with the least position at the 4th digit. Then, if z ends in a 1, we get $\xi = w01110 = w10010$, for some w, which ends in precisely one 0. So assume that z ends in a 0. Then, for some w, $\xi = w0102 = w1000$, so that, by Proposition 1, ξ ends in an odd number of 0s. In conclusion, the first item holds.

For the second item, notice that ξ will end in zero 0s unless, in the subtraction, digit 2 gets a carry and (we may assume that X is Zeckendorf coded) X ends in 3 or more 0s. By assumption this has to be 4 or more 0s. But then, we do not need to add 'a carry' to the second digit in the subtraction. ■

Notice that the second item does not hold if we exchange 'strictly positive' for 'non-negative'. But, as will become apparent, it is only the first item of Lemma 2 which is needed in the proof of Theorem 2.

Lemma 3. *Let $n \in \mathbb{N}$ and suppose that the integer $0 < X < F_{2n-1}$ ends in an even number of 0s, but not in $10^{2t+1}1$, $t \in \mathbb{N}_0$. Then*

$$\varphi := F_{2n-1} - X$$

ends in an odd number of 0s.

Suppose that $T \in \mathbb{N}$ ends in $10^{2t+1}1$ and X in $10^{2s+1}10$, where $T > X$ and $s, t \in \mathbb{N}_0$. Then

$$\xi := T - X$$

ends in an odd number of 0s.

Proof. For the first part, there are two cases to investigate, X is either of the form

(i) $x010^{2t}$, or
(ii) $x010^{2t}1$,

for some $t \in \mathbb{N}$.

Case (i): We have that $F_{2n-1} = 10^{2(n-1)} = (10)^s110^{2t}$, where

$$n - 2 = s + t. \tag{3}$$

Then

$$\varphi = F_{2n-1} - X = (10)^s110^{2t} - x010^{2t} = r10^{2t+1},$$

(where $r = (10)^s0^{2(t+1)} - x0^{2(t+1)} > 0$) which, by Proposition 1 ends in an odd number of 0s independent of r.

Case (ii): We are going to prove that φ ends in precisely one 0. By (3), $F_{2n-1} = (10)^{n-2}11$. Then $\varphi = (10)^{n-2}11 - x010^{2t}1 = r000(10)^t$, which, by $t > 0$, clearly ends in one 0, independent of r.

For the second part, as a first observation, if T is of the form $y0101$, notice that, in Fibonacci numeration, $t \geq 0$ implies

$$\begin{aligned}\xi &= T - X \\ &= y0101 - x010^{2t+1}10 \\ &= y0012 - x010^{2t+1}10 \\ &= z002 \\ &= z010,\end{aligned}$$

for some bit-string z. This idea generalizes to

$$\begin{aligned}\xi &= y010^{2s+1}1 - x010^{2t+1}10 \\ &= y00(10)^s12 - x010^{2t+1}10 \\ &= z010,\end{aligned}$$

where, by assumption, $s \geq 0$. Hence ξ ends in precisely one 0 which resolves the second part of the lemma. ■

Suppose that the ordered pair (X, Y) is of one of the forms in Theorem 2 (ii) to (viii). Then Lemma 2 and 3 together imply that, if any of its option is P then it has to be of the form

$$(V_i, W_i) := (X, Y) \ominus (B_i, A_i). \tag{4}$$

(In other words, if $(X, Y) \ominus (A_i, B_i)$ is a legal option, it is N.)

But, by Theorem 1 (iii), this is impossible if

$$\frac{W_i}{V_i} > \phi.$$

Hence, it suffices to investigate the cases

$$\frac{W_i}{V_i} < \phi. \tag{5}$$

where, by Corollary 1, both W_i and V_i end in an even number of 0s.

Before we prove Theorem 2, let us state a conjecture of 'how far' we believe it could be extended by methods similar to those we have used in the Lemmas and below. (See also Figure 1 and Table 1.)

Conjecture 1. *For all $n \geq 3$ and all i such that*

- $A_i + B_i \leq F_{2n-4}$, *the position* $(F_{2n-1}, F_{2n} - 1 - A_i - B_i)$ *is* P.
- $A_i \leq F_{2n-4}$, *the position* $(F_{2n-1} + A_i, F_{2n} - 1)$ *is* P.

The following elementary result is an important tool for the proof of Theorem 2.

Lemma 4. *For all $n \in \mathbb{N}$,*

$$\frac{F_{2n} - r}{F_{2n-1} - s} > \phi \tag{6}$$

if $0 \leq r \leq \phi s$.

For all $n \in \mathbb{N}_0$,

$$\frac{F_{2n+1} - r}{F_{2n} - s} < \phi \tag{7}$$

if $0 \leq \phi s \leq r$.

Proof. Notice that (6) follows from, for all $n > 0$,

$$\frac{F_{2n}}{F_{2n-1}} > \phi \tag{8}$$

and (7) from, for all $n \geq 0$,

$$\frac{F_{2n+1}}{F_{2n}} < \phi, \tag{9}$$

But (8) and (9) are easy. We give two alternative proofs. Clearly $\frac{F_1}{F_0} < \phi$. If $F_{2n+1} < \phi F_{2n}$ then $F_{2n+1} < \phi(F_{2n+2} - F_{2n+1})$ so that $F_{2n+1}\phi < F_{2n+2}$. If $F_{2n} > \phi F_{2n-1}$ then $F_{2n} > \phi(F_{2n+1} - F_{2n})$ so that $F_{2n}\phi > F_{2n+1}$.

Another proof is given by using the well-known closed form expression $F_n = \frac{\phi^{n+1}-(1-\phi)^{n+1}}{\sqrt{5}}$. By this formula we get

$$\begin{aligned}\frac{F_n}{F_{n-1}} &= \frac{\phi^{n+1} - (1-\phi)^{n+1}}{\phi^n - (1-\phi)^n} \\ &= \phi\frac{\phi^{2n} - (-1)^{n+1}}{\phi^{2n} - (-1)^n},\end{aligned}$$

which gives the claim. □

Proof of Theorem 2. For case (i) we need to prove that $(F_{2n} - 1, F_{2n} - 1)$ only has N-positions as options. By Lemma 1, 2 and 3, for all i,

$$F_{2n} - 1 - b_i \tag{10}$$

ends in an odd number of 0s and hence, by Proposition 1, all options of the form $(F_{2n} - 1, F_{2n} - 1) \ominus (A_i, B_i)$ are N.

For case (ii), by (10), we only need to be concerned with options of the form $(X, Y) := (F_{2n-1}, F_{2n} - 1) \ominus (B_i, A_i)$. Notice that, by the first part of Lemma 4, for all $i \in \mathbb{N}$, $r = A_i + 1 \leq B_i = s$, we get

$$\frac{Y}{X} = \frac{F_{2n} - r}{F_{2n-1} - s} > \phi. \tag{11}$$

Then Theorem 1 (iii) gives the claim.

Case (iii). Here we want to prove that $(X, Y) := (F_{2n-1}, F_{2n} - 4)$ is P. The only move of the form (b_i, a_i) which satisfies (7) in Lemma 4 is $(r, s) = (B_1, A_1) = (2, 1)$. By Theorem 1 (iii) it then suffices to prove that $(X, Y) \ominus (2, 1)$ is N. In fact, we are going to demonstrate that $(X, Y) \ominus (2, 1) = (F_{2n-1} - 2, F_{2n} - 5)$ has $(3, 3)$, which is P (see also Table 1), as an option. For the latter it suffices to verify that

$$\begin{aligned}M &:= (X, Y) \ominus (2, 1) \ominus (100, 100) \\ &= (10^{2t} - 10, 10^{2t+1} - 1000) \ominus (100, 100) \\ &= (10^{2t} - 110, 10^{2t+1} - 1100) \end{aligned}\tag{12}$$

is a legal move (where $t = n - 1$).

Notice that, for $t \geq 2$,

$$10^{2t} = (10)^{t-3}100210$$

and

$$10^{2t+1} = (10)^{t-3}1002100.$$

By inserting these two identities into (12) we get that M is of the form $(z100, z1000)$ and hence legal.

Case (iv). By inspection, we have that

$$\frac{F_{2n-1} - B_i}{F_{2n} - 9 - A_i} > \phi,$$

for all $i \geq 4$. So, by Lemma 4, it suffices to verify that each option

$$(F_{2n-1}, F_{2n} - 9) \ominus \{(2,1), (5,3), (7,4)\}$$

is N, respectively. Hence, it suffices to demonstrate that
A: $(F_{2n-1} - 2, F_{2n} - 10)$ has the option $(6, 6)$,
B: $(F_{2n-1} - 5, F_{2n} - 12)$ has the option $(F_{2n-2}, F_{2n-1} - 4)$, and
C: $(F_{2n-1} - 7, F_{2n} - 13)$ has the option $(3, 3)$,
where the second item follows by case (ii). It suffices to verify that the moves are of the form in Lemma 1.

Item A: We demonstrate that this is a legal move (using Fibonacci coding),

$$\begin{aligned}
&(10^{2n}, (10)^{n-2}00101) \ominus (10, 1) \ominus (10100, 10100)\\
&= (10^{2n}, (10)^{n-2}00101) \ominus (100000, 10101)\\
&= ((10)^{n-3}110000, (10)^{n-3}0110000) \ominus (100000, 10000)\\
&= ((10)^{n-3}010000, (10)^{n-3}0100000).
\end{aligned}$$

Item B: Put $s = n - 2$. Then, the move is

$$\begin{aligned}
&(10^{2s+2} - 1000, 10^{2s+3} - 10101) \ominus (10^{2s+1}, 10^{2s+2} - 101)\\
&= (110^{2s} - 1000, 110^{2s+1} - 10000) \ominus (10^{2s+1}, 10^{2s+2})\\
&= (10^{2s} - 1000, 10^{2s+1} - 10000)\\
&= ((10)^{s-2}1100 - 1000, (10)^{s-2}11000 - 10000)\\
&= ((10)^{s-2}0100, (10)^{s-2}01000),
\end{aligned}$$

which is legal.

Item C: This is similar to item A:

$$\begin{aligned}
&(10^{2n}, (10)^{n-2}00101) \ominus (1010, 101) \ominus (100, 100)\\
&= ((10)^{n-3}012011, (10)^{n-3}0101112) \ominus (10010, 1010)\\
&= ((10)^{n-3}010100, (10)^{n-3}0101000).
\end{aligned}$$

Case (v). By Lemma 4 (as in Case (iii)), it suffices to demonstrate that

$$(F_{2n-2} + 1, F_{2n-1} - 1) \ominus (2, 1) = (F_{2n-2} - 1, F_{2n-1} - 2)$$

is of the form in Corollary 1. But this hold since $F_{2n-2}-1$ is of the form $10^{2t}-1 = 1010\ldots1011-1 = 1010\ldots1010$ which ends in precisely one 0.

Case (vi). By case (ii), positions of the form $(F_{2n-3}, F_{2n-2}-1)$ is P. This also holds for $(11, 11)$. Then one needs to verify that

$$(F_{2n-1}+3, F_{2n}-1) \ominus (1,2) \ominus (11,11)$$

and

$$(F_{2n-1}+3, F_{2n}-1) \ominus (3,5) \ominus (F_{2n-3}, F_{2n-2}-1)$$

are legal moves. We omit the details, since the methods are repetitions of the above. This suffice to prove the claim.

Case (vii). Let us demonstrate that the only options of $(F_{2n-1}+4, F_{2n}-1)$ are of the form in Corollary 1, that is, at least one of the coordinates ends in an odd number of 0s. By Lemma 4, we only need to check the moves $(1,2)$, $(3,5)$ and $(4,7)$. In Fibonacci coding we have that $F_{2n}-1 = 1010\ldots10101 = 1010\ldots01201 = 1010\ldots01112$. But then, by subtracting with $10, 1000$ and 1010 respectively (and using the rule (2)) we are done with this case.

Case (viii). By Lemma 4, here it suffices to verify that each one of the four options $(F_{2n-1}+6, F_{2n}-1) \ominus \{(2,1),(5,3),(7,4),(10,6)\}$ is N. We leave out much of the details since the verifications are repetitions of the above. However, a 'rough line' goes as follows:

It may be verified that $(F_{2n-1}+6, F_{2n}-1) \ominus (1,2)$ has the P-position $(14, 14)$ as an option. By Case (vi), $(F_{2n-3}+3, F_{2n-2}-1)$ is P. This position is an option of $(F_{2n-1}+6, F_{2n}-1) \ominus (3,5)$. The option $(F_{2n-1}+6, F_{2n}-1) \ominus (4,7)$ is N. This follows by Corollary 1, since $10^{2t}+10$ ends in an odd number of 0s. Finally, it may be verified that $(F_{2n-1}+6, F_{2n}-1) \ominus (6,10)$ has the P-position $(1,1)$ as an option. ■

The terminal positions of $W^\star$ are all positions of the form $(0, n), n \in \mathbb{N}$. Denote the non-terminal P-positions of $W^\star$ (with $a_i \le b_i$) by, in lexicographic order, $(a_1, b_1), (a_2, b_2), \ldots$

Corollary 2. *For $n \in \mathbb{N}$, $f(n) := \frac{b_n}{a_n}$ does not converge as $n \to \infty$. In particular, for all $\epsilon \in \mathbb{R}$ and $n \in \mathbb{Z}_{>0}$ there is an $i \ge n$ such that $f(i) = 1$ and a $j = j(\epsilon) \ge n$ such that $\phi - \epsilon < f(j) < \phi$.*

Proof. This follows from Theorem 2 (i) and (ii). In particular, notice that, for all $n \in \mathbb{N}$, (F_{2n-1}, F_{2n}) is a P-position of Wythoff Nim. ■

Question 1. *From Theorem 2 it follows that we may characterize some P-positions of $W^\star$ in polynomial time. Is there any method to extend these results to a polynomial time algorithm of determining if an arbitrary position is P?*

Numerical data, via computer simulations, motivate the following conjecture:

Conjecture 2. *Define the sets*

$S_1 := \{3, 8, 11, 21, 32\}$,

$S_2 := \{129, 362\}$,

$S_3 := \{x \in \mathbb{N} \setminus \{19\} \mid$ *The Zeckendorf coding of* x *ends in 101001*$\}$,

$S_4 := \{x \in \mathbb{N} \mid$ *The Zeckendorf coding of* x *ends in 1*$\}$.

Then, the position (i, i) *belongs to* $\mathcal{P}(W^\star)$ *if* i *belongs to* $(S_1 \cup S_4) \setminus (S_2 \cup S_3)$. *It belongs to* $\mathcal{N}(W^\star)$ *if* i *belongs to* $\mathbb{N} \setminus (S_1 \cup S_2 \cup S_3)$.

Let us give, in order of appearance, the Zeckendorf coding of the numbers in Conjecture 2 (they seem to have some special relevance to $W^\star$ which I do not yet understand).

Remark 1. *The Zeckendorf coding of*

- $3, 8, 11, 21$ *and* 32 *are* $100, 10000, 10100, 1000000$ *and* 1010100 *respectively,*
- 129 *and* 362 *are* 1010001001 *and* 101010001001 *respectively,*
- 19 *is* 101001.

4. Permutation games and the ⋆-operator

We have defined the ⋆-operator in Section 1.1. Suppose that (a_i) and (b_i) are complementary sequences, both increasing. Define G by setting $\mathcal{M}(G) := \{\{a_n, b_n\} \mid n \in \mathbb{N}\}$. The Main Theorem in [LHF] gives sufficient conditions on a and b (for example they may denote any pair of complementary Beatty sequences) such that

$$\mathcal{P}(G^\star) = \mathcal{M}(G) \cup \{\mathbf{0}\} \tag{13}$$

and therefore,

$$G^{\star\star} = G. \tag{14}$$

The question to try and classify all invariant subtraction games G for which (13) and (14) hold is left open. We will here ask a related, but more general question. Let us first explain what we mean by 'convergence' of games.

For $n \in \mathbb{N}_0$ and G an invariant subtraction game, denote

$$G_n = M(G) \cap \{(x, y) | x \leq n\}.$$

Let $(G(k))$ denote a sequence of invariant subtraction games.

Suppose that there is a game H such that, for all n, there is a k such that $H_n = G_n(i)$ for all $i > k$. Then $(G(k))$ converges and hence we can define the limit game $H = \lim_{k \in \mathbb{N}} G(k)$.

Question 2. *Let* G *be an invariant subtraction game. Is it then true that the game*

$$H = \lim_{k \in \mathbb{N}} G^{2k}$$

exists?

Clearly the answer to this question is affirmative for each game which satisfies (13). But this is trivial, so we want to set out to try and find a larger family of games with an affirmative answer to Question 2, but for which, in general, $G \neq G^{\star\star}$. We have defined our candidates, the permutation games, in Section 1.2.

The first result is that the set of all permutation (involution) games is closed under the operation $\star\star$.

Theorem 3. *If G is a permutation game then, so is $G^{\star\star}$. Furthermore, if G is an involution game, so is $G^{\star\star}$.*

Then we give and affirmative answer of Question 2 for G a permutation game.

Theorem 4. *Let G be a permutation game. Then $H = \lim_{k\in\mathbb{N}} G^{2k}$ exists. Furthermore, H is a permutation game. If G is an involution game, so is H.*

Remark 2. *The involution games generalize the games defined by complementary sequences of integers discussed in* [LHF]*. Suppose that $a = (a_i)$ and $b = (b_i)$ are two complementary sequences. Define G by $\mathcal{M}(G) = \{\{a_i, b_i\}\}$. Then G is an involution game. Let $G =$ 2-pile Nim. Then $G^\star$ is an involution game, but $\mathcal{M}(G^\star) = \{\{a_i, b_i\}\}$ gives $a_i = b_i$, for all i, so that a and b are not complementary.*

Remark 3. *The operator $\star\star$ may turn an invariant subtraction game which is not a permutation (involution) game into a permutation (involution) game. For example, define G by $\mathcal{M}(G) = \{(i,i) \mid i \in \mathbb{N}\} \cup \{(1,2)\}$. Then, since $\mathcal{P}(G) = \mathcal{M}(\text{2-pile Nim}) \cup \{\mathbf{0}\}$, we get that $\mathcal{M}(G^{\star\star}) = \{(i,i) \mid i \in \mathbb{N}\}$.*

The next observation is proved in [LHF, Lemma 2.2]. Note here the importance of the word 'invariant'.

Lemma 5 ([LHF])**.** *A move in an invariant subtraction game can never be a P-position*[4]*.*

Lemma 6. *Suppose that G is an invariant subtraction game on $\mathcal{B}(G) = \mathbb{N}_0 \times \mathbb{N}_0$ which satisfies $\{\{0,x\} \mid x \in \mathbb{N}_0\} \subset \mathcal{P}(G)$. Then*

(i) no two P-positions of $G^\star$ lie in the same row or column,
(ii) $\{\{0,x\} \mid x \in \mathbb{N}_0\} \subset \mathcal{P}(G^{\star\star})$.

Proof. Case (i) is obvious since $\mathcal{M}(G^\star) = \mathcal{P}(G)\backslash\{\mathbf{0}\}$. For (ii) we apply the definition of $\star$ twice together with Lemma 5. Namely, the assumption

$$\{\{0,x\} \mid x \in \mathbb{N}\} \subset \mathcal{P}(G)\backslash\{\mathbf{0}\} = \mathcal{M}(G^\star)$$

implies

$$\{\{0,x\} \mid x \in \mathbb{N}\} \cap \mathcal{P}(G^\star) = \{\{0,x\} \mid x \in \mathbb{N}_0\} \cap \mathcal{M}(G^{\star\star}) = \emptyset.$$

Then each $(x,y) \in \mathcal{M}(G^{\star\star})$ satisfies $x > 0$ and $y > 0$. This gives that the set $\{\{0,x\} \mid x \in \mathbb{N}_0\}$ is a subset of all terminal P-positions of $G^{\star\star}$, which is (ii). ■

Since a permutation game satisfies the conditions of Lemma 6, we get the following corollary.

[4] This requires normal play, see also Section 5.6.

Corollary 3. *Let G be a permutation game. Then $\{\{0, x\} \mid x \in \mathbb{N}_0\} \subset \mathcal{P}(G)$ and $\{\{0, x\} \mid x \in \mathbb{N}_0\} \subset \mathcal{P}(G^{\star\star})$.*

The next observation relaxes the requirements in Theorem 1 (iii).

Lemma 7. *Suppose that G is an invariant subtraction game satisfying the assumptions in Lemma 6, for example a permutation game. Suppose further that the column x contains the move (x, y). Then (x, z) is N if $z > y$.*

Proof. By the assumption, $z > y$ implies that

$$(0, z - y) = (x, z) \ominus (x, y)$$

is P. ■

Proof of Theorem 3. Suppose that G is a permutation game. We are going to show that the same holds for $G^{\star\star}$. First we demonstrate that, if there is a move in a non-zero column, then it is unique. Then we demonstrate that each non-zero column contains at least one move. (The arguments obviously work fine with 'column' exchanged for 'row'.) At last we prove the claim of 'symmetry' in case G is an involution game.

Uniqueness: By the definition of a permutation game, no position of the form $\{0, x\}$ is a move in G. Hence all positions of this form belong to $\mathcal{P}(G) = \mathcal{M}(G^\star) \cup \{\mathbf{0}\}$. But (by Lemma 6 (i)) this gives that there can be no two positions of $\mathcal{P}(G^\star) = \mathcal{M}(G^{\star\star}) \cup \{\mathbf{0}\}$ in the same row or in the same column.

Existence: Suppose that there is a least column, say $x_0 > 0$ which does not contain a move of $G^{\star\star}$, that is $\mathcal{M}(G^{\star\star}) \cap \{(x_0, t) \mid t \in \mathbb{N}_0\} = \emptyset$ so that $\{(x_0, t) \mid t \in \mathbb{N}_0\} \subset \mathcal{N}(G^\star)$. Then, by definition of P and N, the set

$$\{(x, y) \mid 0 < x < x_0\} \cap (\mathcal{M}(G^\star) \cup \mathcal{P}(G^\star))$$

must be infinite. By the proof of '*Uniqueness*' we already know that $\mathcal{P}(G^\star) \cap \{(x, y) \mid 0 < x < x_0\}$ is finite, which leaves us with an infinite set $\mathcal{M}(G^\star) \cap \{(x, y) \mid 0 < x < x_0\}$. But, since G is a perumtation game, we can use Lemma 7 on each column implying a finite set $\mathcal{P}(G) \cap \{(x, y) \mid 0 < x < x_0\}$. But then, by the definition of $\star$, this is the desired contradiction, so that there cannot exist such an x_0.

Symmetry: If G is an involution game it follows that also $\mathcal{M}(G^\star) = \mathcal{P}(G) \setminus \{\mathbf{0}\}$ is symmetric, which implies that $\mathcal{M}(G^{\star\star}) = \mathcal{P}(G^\star) \setminus \{\mathbf{0}\}$ is symmetric. But, by the first part, we already know that $G^{\star\star}$ is a permutation game, hence also an involution game. ■

Proof of Theorem 4. Fix a permutation game G. Then, by induction on Theorem 3, we get that, for all $k \in \mathbb{N}_0$, G^{2k} is also. Suppose now that, for a fixed $k \in \mathbb{N}$, there is a least column $x_0 > 0$ such that

$$(x_0, y) \in \mathcal{M}(G^{2k}) \tag{15}$$

and

$$(x_0, z) \in \mathcal{M}(G^{2k+2}), \tag{16}$$

but $y \neq z$. (If there is no such k then we are trivially done.) Then

Claim 1: $(x_0, z) \in \mathcal{N}(G^{2k})$.

Claim 2: For all $r < x_0$, $(r, s) \in \mathcal{P}(G^{2k})$ if and only if $(r, s) \in \mathcal{P}(G^{2k+2})$.

Claim 3: $z > y$.

Claim 4: $(x_0, z) \in \mathcal{M}(G^{2k+4})$.

Proof of Claim 1-4. Claim 1: Suppose on the contrary that $(x_0, z) \in \mathcal{P}(G^{2k})$. Then $(x_0, z) \in \mathcal{M}(G^{2k+1})$, which (by [LHF, Lemma 2.2]) implies $(x_0, z) \in \mathcal{N}(G^{2k+1})$, which, by definition of $\star$, contradicts the assumption (16).

Claim 2: Since, by assumption, the only move that differs in the two games is in column x_0 or greater, the claim follows.

Claim 3: By Claim 1, $(x_0, z) \in \mathcal{N}(G^{2k})$. Suppose that $z < y$. Then, by (15) and the definition of a permutation game, there must exist a move, say $(r, s) \in \mathcal{M}(G^{2k})$, with $0 < r < x_0$ and $0 < s < z$ such that the option

$$\begin{aligned}(X, Y) &:= (x_0, z) \ominus (r, s) \\ &\in \mathcal{P}(G^{2k}) = \mathcal{M}(G^{2k+1}) \cup \{\mathbf{0}\}. \end{aligned} \tag{17}$$

By minimality of x_0, we also have

$$(r, s) \in \mathcal{M}(G^{2k+2}),$$

which, by definition of $\star$, implies

$$(r, s) \in \mathcal{P}(G^{2k+1}).$$

This, together with (17), gives that

$$\begin{aligned}(x_0, z) &= (X, Y) \oplus (r, s) \\ &\in \mathcal{N}(G^{2k+1}),\end{aligned}$$

which, by definition of $\star$, contradicts the assumption (16). The claim follows.

Claim 4: Suppose this does not hold. Then, by Claim 3 and since, by Theorem 3, G^{2k+4} is a permutation game, there is a $w > z$ such that

$$(x_0, w) \in \mathcal{M}(G^{2k+4}) \tag{18}$$

and it is unique. By definition of $\star$, $(x_0, z) \in \mathcal{P}(G^{2k+1})$. Then, by definition of $\mathcal{P}$ and $\mathcal{M}$ in an invariant subtraction game, it is clear that, in Claim 3, z is the least number such that, for all

$$(r, s) \in \mathcal{M}(G^{2k+1}) \tag{19}$$

we have that

$$(x_0, z) \ominus (r, s) \in \mathcal{N}(G^{2k+1}).$$

Observe that, by the uniqueness of (x_0, w) in (18), we get that $(x_o, z) \in \mathcal{N}(G^{2k+3})$. Then, by using, in essence, the same argument as in Claim 2 and 3, the P-positions of the games G^{2k+1} and G^{2k+3} must coincide in the columns below x_0. This gives that there has to be a move $(u,v) \in \mathcal{M}(G^{2k+3})$ *not* of the form in (19), such that

$$(x_0, z) \ominus (u,v) \in \mathcal{P}(G^{2k+3}). \tag{20}$$

But then, by definition of $\star$,

$$(u,v) \in \mathcal{P}(G^{2k+2}) \setminus \{\mathbf{0}\}, \tag{21}$$

so that, by Claim 2, $(u,v) \in \mathcal{P}(G^{2k}) \setminus \{\mathbf{0}\} = \mathcal{M}(G^{2k+1})$. But then (19) together with the definition of (u,v) gives the desired contradiction. Since, by Theorem 3, G^{2k+4} is a permutation game we are done with this case.

Again, by Theorem 3, G^{2k} is a permutation game for all $k \in \mathbb{N}_0$. Together with Claim 4, this gives the existence of the permutation game H.

If G is an involution game, then, by Theorem 3, for all $k \in \mathbb{N}_0$, G^{2k} is. This gives that H is also an involution game. ∎

5. Ornament games and complementary Beatty sequences with rational moduli

In [Fr08] the 'variant' game 'The Mouse game' is introduced. It answered the question:

Question 3. *Is there an impartial game G with the set of P-positions defined by complementary Beatty sequences with rational moduli?*

The set of P-positions of the Mouse game is

$$S := \{\{a_i, b_i\} \mid i \in \mathbb{N}\} \cup \{\mathbf{0}\}, \tag{22}$$

where

$$a_i := \left\lfloor \frac{3i}{2} \right\rfloor \text{ and } b_i := 3i - 1. \tag{23}$$

The Mouse game is an extension of Wythoff Nim, where the available moves depend on which particular position of the board the next player moves from. (It is 'variant', because it is not invariant.) Precisely, if the position is (x,y), where $y - x \equiv 0 \pmod 3$, then a player may move

$$(x,y) \to (w,z),$$

provided $x - w \geq 0$, $y - z \geq 0$ and

$$\mid (x-w) - (y-z) \mid \leq 1.$$

Otherwise the moves are as in Wythoff Nim.

In [LHF] we gave an affirmative answer to the following question:

Question 4. *Is there an invariant subtraction game G with the set of (non-zero) P-positions defined by a pair of complementary Beatty sequences with rational moduli?*

However we did not provide any closed form of the moves of any such game. If, in Question 4, 'rational' is exchanged for 'irrational' the game of Wythoff Nim provided a solution over 100 years ago.

5.1. **The invariant moves of the Mouse trap.** Here we study an invariant subtraction game, *the Mouse trap* = (the Mouse game)**, introduced in [LHF], with (by [LHF, Main Theorem]) an identical set of P-positions as the Mouse game. See Figure 4. We present a closed form for the moves of this game and give an affirmative answer to this question.

Question 5. *Is there an invariant subtraction game G with the set of P-positions defined by complementary Beatty sequences with rational moduli and with the set of moves given by a polynomial time algorithm?*

In fact, in Section 5.3 we give an affirmative answer to the same question, but this time for an infinite family of games. But the Mouse trap has an interest for its own sake. Here we give its invariant moves.

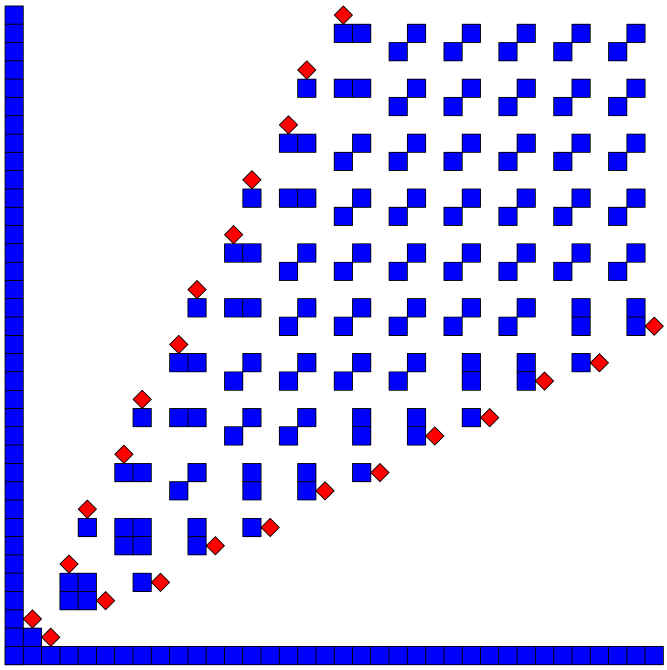

FIGURE 3. The red diamonds and the blue squares represent the initial moves and P-positions of (Mouse game)* respectively. Hence the blue squares, except $\mathbf{0}$, symbolizes the moves of 'the Mouse trap' defined in Theorem 5. The corresponding P-positions are then the red diamonds together with $\mathbf{0}$.

Theorem 5. *With notation as in (23), define*

$$\mathcal{M}_0 = \{(1,1),(3,3),\{3,4\},(4,4),\{4,7\},(6,6),\{6,7\},\{6,10\}\}$$

$$\mathcal{M}_1 = \{\{a_{2n-1},a_{2m-1}\} \mid m,n \in \mathbb{N}, 3 \le n \le m < 2n-1\}$$

$$\mathcal{M}_2 = \{\{a_{2n},a_{2m}\} \mid m,n \in \mathbb{N}, 3 \le n \le m < 2n-2\}$$

$$\mathcal{M}_3 = \{\{a_{2n},a_{4n-1}\},\{a_{2n},a_{4n-3}\} \mid 3 \le n \in \mathbb{N}\}$$

$$\mathcal{M}_4 = \{\{0,x\} \mid x \in \mathbb{N}\}.$$

Define $G =$'the Mouse trap' by

$$\mathcal{M}(G) = \bigcup_{i=0}^{4} \mathcal{M}_i. \tag{24}$$

Then, with notation as in (22),

$$\mathcal{P}(G) = S.$$

Proof. Notice that, by (23), for all $n \in \mathbb{N}$,

$$a_{2n} = 3n, \tag{25}$$

$$a_{2n-1} = 3n - 2 \tag{26}$$

and

$$b_n = 3n - 1. \tag{27}$$

It is easy to verify that, given the moves in $\mathcal{M}_0$, we get that $(a_1, b_1) = (1, 2), (a_2, b_2) = (3, 5), (a_3, b_3) = (4, 8)$ and $(a_4, b_4) = (6, 11)$ are the unique P-positions above the main diagonal up to and including column 6. (See also Figure 4.) So assume that the column is ≥ 7. Let us begin with the direction

$N \to P$: Let $(X, Y) \notin S$, with say $X \leq Y$. Then we need to prove that (X, Y) has an option in S. For three distinct classes of positions this is already clear, namely if

- $X = b_i$, some i,
- $X = a_i$ and $Y > b_i$, some i,
- $(X, Y) \in \mathcal{M}(G)$.

The first two items follow immediately from the definition of $\mathcal{M}_4$ (see also [LHF, Main theorem], Theorem 1). The third item is immediate by the definition of an invariant move, namely $(X, Y) \ominus (X, Y) = \mathbf{0}$, which (by normal play) is P. (See also Lemma 5.) So assume that (X, Y) does not belong to any of these three classes of positions. Then, by the first two items,

$$(X, Y) = (a_i, a_j) \tag{28}$$

for some $i \leq j$ and with $a_j < b_i$. But also, by the third item, $\mathcal{M}_1$ and $\mathcal{M}_2$, we have that

$$i \not\equiv j \pmod 2, \tag{29}$$

or equivalently

$$a_i \not\equiv a_j \pmod 3, \tag{30}$$

except, by definition of $\mathcal{M}_2$ and $\mathcal{M}_3$, if both i and j are even and $j \geq 2i - 4$, in which case (X, Y) is of the form

$$(a_{2n}, a_{4n-2}) \text{ or } (a_{2n}, a_{4n-4}). \tag{31}$$

Altogether, we claim that:

Claim 1: If $X \equiv 0 \pmod 3$ and $Y \equiv 1 \pmod 3$ then the next player can move to the position $(5, 3) \in S$.

Claim 2: If $X \equiv 1 \pmod 3$ and $Y \equiv 0 \pmod 3$ then the next player can move to the position $(3,5) \in S$.

Claim 3: If (31) holds then the next player can move to the position $(3,5) \in S$.

By Figure 4 it is not hard to justify the columns 7 to 11. For columns greater than 11, we begin with

Proof of Claim 1: We have that $(X,Y) = (3n+6, 3m+7) \ominus (5,3) = (3n+1, 3m+4) \in \mathcal{M}_1$ if and only if $(X,Y) \prec (a_i, b_i)$, some i. We will demonstrate that the 'worst possible case' gives $(X,Y) \ominus (5,3) = (a_i, b_i - 1)$, some i. This happens whenever Y/X is 'maximized', that is, by definition of $\mathcal{M}_2$ and $\mathcal{M}_3$, whenever

$$\begin{aligned}(X,Y) &= (a_{2n}, a_{4n-5})\\ &= (a_{2n}, a_{2(2n-2)-1})\\ &= (3n, 3(2n-2)-2)\\ &= (3n, 6n-8).\end{aligned}$$

Hence, we get

$$\begin{aligned}(X,Y) \ominus (5,3) &= (3n, 6n-8) \ominus (5,3)\\ &= (3(n-1)-2, 6(n-1)-5)\\ &= (a_{2(n-1)-1}, b_{2(n-1)-1} - 1),\end{aligned}$$

as claimed. Here we have made repeated use of (25), (26) and (27).

Proof of Claim 2: $(3n+4, 3m+6) \ominus (3,5) = (3n+1, 3m+1) \in \mathcal{M}_1$ if and only if $(X,Y) \prec (a_i, b_i)$, some i. As in Claim 1, the worst possible case gives $(X,Y) \ominus (5,3) = (a_i, b_i - 1)$. We omit the details.

Proof of Claim 3: $(3n+3, 3m+6) \ominus (3,5) = (3n, 3m+1) \in \mathcal{M}_3$, We have either $(a_{2n}, a_{4n-2}) \ominus (3,5)$ or $(a_{2n}, a_{4n-4}) \ominus (3,5)$. The first case is equivalent to $(3n, 3(2n-1)) \ominus (3,5) = (3(n-1), 6(n-1)-2) = (a_{2(n-1)}, b_{2(n-1)} - 1) \in \mathcal{M}_3$, since $a_{4n-1} = 6n-2 = b_{2n} - 1$. The second case is treated in analogy to this.

$P \to N$: Assume $(X,Y) = (a_i, b_i)$, some i. It will become apparent that also for this case it suffices to analyze a 'worst case scenario'. This is when the move is of the form $(a_j, b_j - 1)$. We get the option

$$(Z,W) := (X,Y) \ominus (a_j, b_j - 1) = (a_i - a_j, b_i - b_j + 1) = (a_i - a_j, 3(i-j)+1).$$

There are four cases to investigate,

(i) $i = 2n, j = 2m,$,
(ii) $i = 2n, j = 2m-1$,
(iii) $i = 2n-1, j = 2m$,
(iv) $i = 2n-1, j = 2m-1$.

where $n, m \in \mathbb{N}$. For both cases (i) and (iv), we get the option

$$(Z, W) = (3(n-m), 6(n-m)+1) \tag{32}$$
$$= (a_{2(n-m)}, b_{2(n-m)} + 2) \tag{33}$$
$$\notin S. \tag{34}$$

Case (ii) gives $3(n-m)+2 = Z \equiv 2 \pmod 3$, which is of the form of a 'b-coordinate'. This gives $(Z, W) \notin S$ since $Z < W = 6(n-m)+4$. For case (iii) we get that

$$\begin{aligned}(Z, W) &= (3(n-m)-2, 6(n-m)-2)\\ &= (a_{2(n-m)-1}, b_{2(n-m)-1} + 2)\\ &\notin S.\end{aligned}$$

By symmetry, we are done. ∎

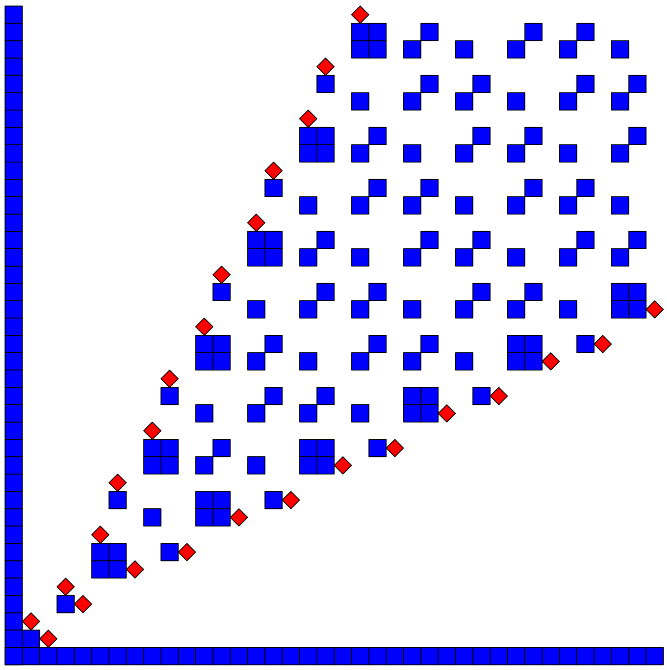

FIGURE 4. An initial view of a close relative to (the Mouse game)*, namely where the complementary Beatty sequences are defined by $\alpha = 2/3, \delta = -1/3, \beta = 1/3, \gamma = 2/3$, notation as in (35) and (36). Notice that there are only three ornament games of Class 3. Thus the Figures 3, 4 and 5 together illustrate this whole class. See also [Pres10] for updates of plots on more classes of ornament games.

Remark 4. *Since the above* (a_i) *and* (b_i) *are complementary Beatty sequences, by* [LHF, Main Theorem]*, the moves defined in Theorem 5 are identical to the non-zero* P*-positions of (Mouse game)***.*

Remark 5. *By Theorem 5, the decision problem for (Mouse game)* has polynomial complexity. In contrast we do not know if this holds for (Wythoff Nim)*, where we, so far, only have the partial results in Corollary 1 and Theorem 2 and together with the conjectures in Conjecture 1 and 2.*

5.2. **Counting rational Beatty sequences and classes of ornament games.** In [O'B03] the author provides a simple proof for the conditions of pairs of Beatty sequences to be complementary. To this purpose the sequences in (1) are translated to the forms

$$\left(\left\lfloor \frac{n-\delta}{\alpha} \right\rfloor\right)_{n\in\mathbb{N}} \tag{35}$$

and

$$\left(\left\lfloor \frac{n-\gamma}{\beta} \right\rfloor\right)_{n\in\mathbb{N}}, \tag{36}$$

$\alpha, \beta, \delta, \gamma \in \mathbb{R}$. Clearly, for density reasons, complementarity implies

$$\alpha + \beta = 1 \tag{37}$$

and we may assume that $\alpha \le \beta$. Hence, if one of the sequences has a rational modulus, then the other has also. Here we consider *the rational case*. Then there is a least integer q such that

$$q\alpha \in \mathbb{N} \tag{38}$$

and by (37), $q > 1$. Then, by [O'B03], the sequences are complementary if and only if

$$\frac{1}{q} \le \alpha + \delta \le 1 \tag{39}$$

and

$$\lceil q\delta \rceil + \lceil q\gamma \rceil = 1. \tag{40}$$

Fix a constant $1 < C \in \mathbb{N}$ and let us estimate the number of pairs of rational Beatty sequences which together satisfy (37), (38), (39), (40) and $q = C$. Let Ξ denote the number of pairs of such sequences. It turns out that Ξ only depends on q (in particular it is independent of the reals δ and γ).

Proposition 2. *Given $C \in \mathbb{N}$, Ξ is finite. $\Xi = C \times (\varphi(C) + 1)/2$, where φ denotes the number of positive integers coprime with and less than C.*

Proof. With notation as above, by definition of $C = q$, there is a $p \in \{1, 2, \dots, q-1\}$ such that $\alpha = \frac{p}{q}$ with $\gcd(p, q) = 1$. Thus, we may rewrite (35) as $(\lfloor \frac{qn-q\delta}{p} \rfloor)_{n\in\mathbb{N}}$ (and (36) as $(\lfloor \frac{qn-q\gamma}{q-p} \rfloor)_{n\in\mathbb{N}}$). For a fixed α, put

$$r_n(\delta) := \left\lfloor \frac{qn - q\delta}{p} \right\rfloor.$$

Claim 1: For $t \in \{-p+1, -p+1, \dots, q-p\}$ and $s \in \mathbb{R}$, $r_n(\frac{t}{q}) < r_n(\frac{s}{q})$ implies that $s \le t - 1$.

Claim 2: For each $n \in \mathbb{N}$, there exists an $m \in \{n, n+1, \dots, n+q-1\}$ such that $r_m(\frac{t}{q}) < r_m(\frac{t-1}{q})$.

Assume that these two claims hold. Then we get that there are precisely q distinct pairs of complementary Beatty sequences satisfying (39) and (40). Namely, we obviously need $\beta = 1 - \alpha$ and with t as in Claim 1, take $\delta = \frac{t}{q}$ and $\gamma = \frac{1-t}{q}$. By symmetry (this is the division by 2), the proposition follows. But we need to prove the claims.

Proof of Claim 1: Suppose that $r_n(\frac{t}{q}) = \lfloor \frac{qn-t}{p} \rfloor < \lfloor \frac{qn-s}{p} \rfloor$. Then $\frac{qn-s}{p} - \frac{qn-t}{p} \geq \frac{1}{p}$ so that $-s + t \geq 1$.

Proof of Claim 2: We have that $\gcd(p, q) = 1$. Then there exists an $m \in \mathbb{N}$ such that $\frac{qm-t+1}{p} \in \mathbb{N}$. This implies that

$$\left\lfloor \frac{qm-t}{p} \right\rfloor < \frac{qm-t}{p} + \frac{1}{p} = \left\lfloor \frac{qm-t+1}{p} \right\rfloor$$

■

Let $(\{\alpha n + \delta)$ and $(\beta n + \gamma)$ denote two complementary sequences with rational moduli. Suppose that the invariant subtraction game G is defined by $\mathcal{M}(G) = \{\{\alpha n + \delta, \beta n + \gamma\} \mid n \in \mathbb{N}\}$. Then we call $G^\star$ an *ornament game* (e.g. Figure 2). If, in addition, $\alpha = \frac{C}{D}$ with $\gcd(C, D) = 1$, then $G^\star$ is an ornament game of *Class C*.

Corollary 4. *The number of distinct ornament games of Class $C \in \mathbb{N}$ is $C \times (\varphi(C) + 1)/2$.*

5.3. **A subsided family of ornament games.** In Section 5.1 we studied a special case of the Class 3 ornament games. With notation as in Theorem 6, the games of the form $G^\star$ makes up a general family of ornament games with precisely one member in each class—thus, for example, the whole of Class 2 which consists of one single game $G^\star$ (with $\mathcal{M}(G) = \{\{2n, 2n-1\} \mid n \in \mathbb{N}\}$ and $\mathcal{M}(G^\star) \cup \mathbf{0} = \mathcal{P}(G) = \{\{2n-1, 2n-1\} \mid n \in \mathbb{N}\} \cup \{\{0, x\} \mid x \in \mathbb{N}\}$). We call $\{G^\star\}$ the family of 'subsided' ornament games (e.g. Figure 5).

Theorem 6. *Let $2 \leq q \in \mathbb{N}$ and, for all $n \in \mathbb{N}$, put*

$$a_n := \left\lfloor \frac{qn-1}{q-1} \right\rfloor, b_n := qn.$$

Define G by

$$\mathcal{M}(G) = \{\{a_n, b_n\} \mid n \in \mathbb{N}\}.$$

Then $\mathcal{P}(G) = S \cup \{\{0, x\} \mid x \in \mathbb{N}_0\}$, where

$$S := \{(qn+s, qn+t) \mid s, t \in \{1, 2, \ldots, q-1\}, n \in \mathbb{N}\}. \tag{41}$$

Proof. Since, for all $n, a_n \leq b_n$ and both sequences are increasing, by Theorem 1 it suffices to study positions of the form (a_i, a_j). By the same theorem and by symmetry, it suffices to study positions of the form (x, y) with $a_n \leq x \leq y < b_n$. By elementary algebra we get that, for $i \in \{0, 1, \ldots, q-2\}$ and $n \in \mathbb{N}$,

$$a_{(q-1)n-i} = qn - i - 1$$

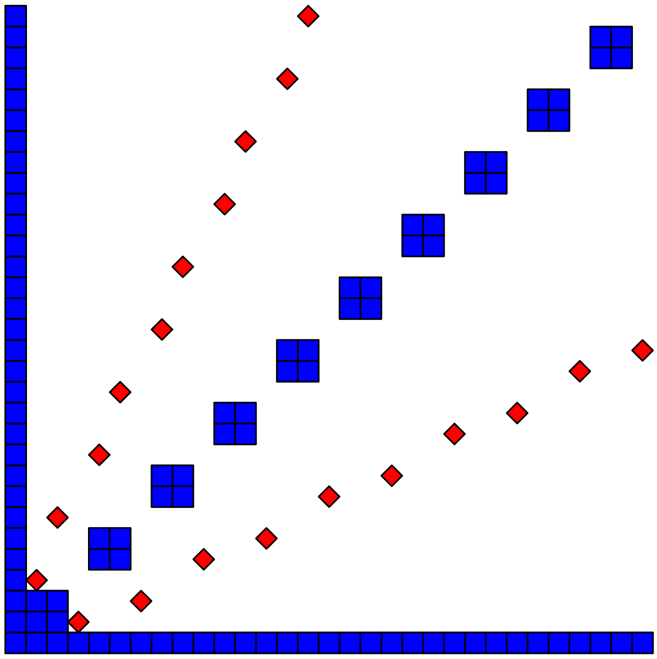

FIGURE 5. An initial view of a close relative to (the Mouse game)*, namely where $\alpha = 2/3, \delta = 1/3, \beta = 1/3, \gamma = 0$, see (35) and (36). The blue squares except **0** may also be viewed as the first few moves of the unique Class 3 game in the family of 'subsided' ornament games, see Theorem 6.

and

$$b_n = qn.$$

Fix $r, s \in \mathbb{N}$ with $r \le s$, $i, j \in \{0, 1, \dots, q-2\}$, and consider the position

$$(X, Y) := \left(a_{(q-1)r-i}, a_{(q-1)s-j}\right) = (qr - i - 1, qs - j - 1)\,.$$

$P \to N$: If $r = s$ then none of the options of (X, Y) is of the form in (41).

$N \to P$: If $r < s$ then there is an option of (X, Y) of the form in (41).

Together, these two claims suffices to prove the theorem. In an attempt to avoid uneccesary technicality we give the rest of the proof for the case $q = 3$. The general case may be treated in analogy[5].

Proof of $P \to N$: For $q = 3$ we get

$$(X, Y) = (3r - i - 1, 3r - j - 1).$$

Since $b_1 = 3$, for the case $r = 1$ we are done since X has no legal options, so assume $r > 1$. What is $(X, Y) \ominus (a_n, b_n)$? It suffices to prove that, for all $0 < n \le r$, $X - a_n \ge 3(r - n)$. (By definition of r, all positions (x, y)

[5]Incidentally, C.L.Bouton used the analog approach of 'reducing technicality' in [Bo02] in proving the famous strategy of q-pile Nim. The case $q = 2$ is 'too special' to be valid as a general proof and the cases $q \ge 3$ are analogous.

of the form in (41) satisfy $x < 3(r-n)$, and $3n$ is the corresponding decrease of Y.) By definition of a_n, we have two cases to consider: $n = 2m$ or $n = 2m-1$, $m \in \mathbb{N}$. Suppose first that $n = 2m$. Then, since $m > 0$, $X - a_n = 3(r-m) - 1 > 3r - 6m = 3(r-n)$. If, on the other hand $n = 2m-1$, then $X - a_n = 3(r-m) \geq 3r - 6m + 3 = 3(r-n)$, with equality if and only if $m = 1$.

Proof of $N \to P$: Here

$$(X, Y) = (qr - i - 1, qs - j - 1),$$

with $r < s$ and $i, j \in \{0, 1\}$. It suffices to demonstrate the existance of a move of the form (a_{2n-k}, b_{2n-k}), $k \in \{0, 1\}$ such that the option

$$(Z, W) := (X, Y) \ominus (a_{2n-k}, b_{2n-k})$$

is contained in the set S. By a simple calculation we get that

$$(Z, W) = (3(r-n) - i + k, 3(s - 2n + k) - j - 1). \tag{42}$$

By the definition of the set S, it is required that $i \neq k$. Hence, we get two cases to consider:

Case $k < i$: (Clearly, for $q = 3$ this forces $k = 0$ and $i = 1$, but in the coming we keep the symbols to make the generalization to $q \geq 3$ more transparent.) Then both $-i + k$ and $-j - 1$ are of the correct form as given in (41). Thus, by (42), it suffices to show that there is an $n \in \mathbb{N}$ satisfying $r - n = s - 2n + k$. By the assumption $s > r$ and $k \in \{0, 1\}$ we may take $n := s - r + k \in \mathbb{N}$.

Case $k > i$: For this case we may define n and k via:

$$3(r-n) = 3(s - 2(n-1) + k) + 3$$

and

$$0 < k - i \leq 3 - j - 1 < 3.$$

The former equation gives $n = s + k - r + 1 \in \mathbb{N}$ and the latter is clear. We are done. ■

Remark 6. *Fix irrational moduli of a pair of complementary Beatty sequences. Then, by varying the real offsets, by* [O'B03], *it is easy to see that the corresponding (the irrational analogous to the ornament games) invariant subtraction games are uncountably many. Further classification of such games is left for future research.*

5.4. **A new blocking maneuver on k-Wythoff Nim.** There is another, somewhat simpler, invariant game with the set of P-positions precisely S as in (22). For this game we define a certain 'blocking maneuver/Muller twist' [HL06, La08] on k-Wythoff Nim [Fr82]. The position **0** does not appear among the pairs in (22). Therefore, this position has been treated with some extra care in the following. Let $k \geq 3$. Define the game 'the constrained k-Mouse' as follows: Given a position $(x, y) \in \mathbb{N}_0 \times \mathbb{N}_0$, move as in k-Wythoff Nim [Fr82], that is a player may move $(x, y) \to (x - i, y - j)$, $0 \leq \mid j - i \mid < k$, with $x \geq i$, $y \geq j$. But, before the next player moves, the previous player is allowed to block off at most $k-2$ positions of the form $(x-i, y-j)$,

$$0 < \mid j-i \mid < k \tag{43}$$

and declare that the next player may not move there. If the terminal position $\mathbf{0}$ is of this form it may be blocked off, irrespective of the number of otherwise blocked off positions. When the next player has moved any blocked options are forgotten.

Theorem 7. *Let G denote the constrained 3-Mouse. Then $\mathcal{P}(G) = S$.*

We omit the proof since it is in analogy to the blocking variations of Wythoff Nim presented in [HL06, La08]. The move rules are considerably less technical for this blocking variation compared to the ones of 'the Mouse trap', both being invariant games. An open question is: What is $\mathcal{P}(G)$ for $k > 3$?

5.5. **A conjecture on periodicity.** Let $S \subset \mathbb{N} \times \mathbb{N}$. Then S is *periodic* if, for all (sufficiently large) $(r, s) \in S$ implies that there is a pair $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$ such that, for all $n \in \mathbb{N}_0$,

$$(r + \alpha n, s + \beta n) \in S. \tag{44}$$

If, in a periodic set S, the number of distinct (α, β):s is bounded by a constant k, we say that S is (at most) k-fold periodic.

Conjecture 3. *Let $(a_n) = (\lfloor \alpha n + \delta \rfloor)$ and $(b_i) = (\lfloor \beta n + \gamma \rfloor)$ denote complementary Beatty sequences, $\alpha, \beta, \delta, \gamma \in \mathbb{R}$, $\alpha, \beta > 0$. Define G via $\mathcal{M}(G) = \{(a_n, b_n) \mid n \in \mathbb{N}\}$, symmetric notation. Then $\mathcal{P}(G)$ is periodic if and only if the modulus α of (a_n) is rational.*

Does this conjecture hold with 'periodic' exchanged for '2-fold periodic'?[6]

5.6. **A question on Misère invariant subtraction games.** In [LHF] (Remark 2) we have stated a belief that pairs of inhomogeneous complementary Beatty sequences might be worthy candidates as P-positions in Misère variations of invariant subtraction games. But having investigated a little further there seems to be some problem in constructing such games. The following example illustrates what can happen. The sequence

$$\{0, 1\}, \{2, 4\}, \{3, 7\}, \{5, 10\}, \ldots$$

constitute the P-positions of the Mouse game minus 1 in each coordinate. Hence the corresponding sequences of increasing integers are complementary Beatty sequences on the *non-negative* integers. A Misère subtraction game requires that $\mathbf{0}$ is N. Then $(0, 1)$ is P if it is a move. But the position $(2, 3)$ has to be a move in a game for which the (complementary) P-positions begin with $(0, 1), (2, 4)$ and $(3, 7)$. This follows since $(3, 3)$ is N and the only P-position $\prec (3, 3)$ is $(0, 1)$, which forces $(3, 3) \ominus (0, 1)$ to be a legal move. Clearly, this 'short-circuits' the P-positions $(2, 4)$ and $(0, 1)$. In this context it should be noted that in Lemma 5 which states that in an invariant

[6] A related question is: Does each invariant subtraction game defined via complementary Beatty sequences with irrational moduli (such as W*) have an infinite number of 'log-periodic' P-positions?

subtraction game, a P-position can never be a move, normal play is required. Otherwise it is not true. Rather, if (X, Y) is a P-position in our above example, in Misère play, the moves

$$(X, Y) \ominus (0, 1) \text{ and } (X, Y) \ominus (1, 0)$$

are forbidden.

Question 6. *Are there Misère variations of invariant subtraction games on* $\mathbb{N}_0 \times \mathbb{N}_0$*, such that the* P*-positions are given precisely by complementary inhomogeneous Beatty sequences of non-negative integers and with* $(0, 1)$ *and* $(1, 0)$ the least P*-positions?*

6. The dual of k-pile Nim

Fix a $k \in \mathbb{N}$ and let $\mathcal{B} = \mathbb{N}_0^k$. In this section we think of $\mathcal{B}$ as k piles of tokens. Let G denote k-pile Nim. Then $\mathcal{M}(G) = \{\{x, 0, 0, \ldots, 0\} \mid x \in \mathbb{N}\}$. It is well-known [Bo02] that $\mathcal{P}(G)$ is the set of all k-tuples with Nim-sum zero. We will now demonstrate that k-pile Nim has a dual game.

Theorem 8. *Let* $k \in \mathbb{N}$ *and let* G *denote* k*-pile Nim. Then* $G^{\star\star} = G$*, that is* $G^\star$ *is the dual game of Nim.*

Proof. Suppose that Alice plays first and Bob second in the game of $G^\star$, that is the allowed moves are all non-zero l-tuples with Nim-sum zero, $2 \leq l \leq k$. Notice that for this game a player must remove tokens from at least two piles. It is then clear that the set of terminal positions is $\mathcal{T}(G^\star) = \mathcal{M}(G) \cup \{\mathbf{0}\} \subseteq \mathcal{P}(G^\star)$. Denote the initial position with $(x_1, x_2, \ldots, x_k)$. We may assume $x_i \in \mathbb{N}$ for all i and $x_1 \leq x_2 \leq \ldots \leq x_k$. Then, a winning strategy for Alice is to, *in one and the same move*,

- remove all x_1 tokens from pile 1 and the same number from pile x_2,
- if $x_2 > x_1$, remove $x_2 - x_1$ tokens from pile 2, that is the remaining ones, and the same number from pile 3, otherwise remove all tokens from pile 3 and x_3 tokens from pile 4,
- continue in this manner until all piles are empty except possibly pile k which now contains $x_k - x_{k-1} + \ldots + x_2 - x_1 \geq 0$ tokens if k is even or $x_k - x_{k-1} + \ldots - x_2 + x_1 \geq 0$ if k is odd.

Alice's move is legal, since the Nim-sum of the number of removed tokens in the respective piles is zero. There is at most one pile of tokens left. Hence Bob cannot move and so Alice wins in her first move. This gives $\mathcal{P}(G^\star) = \mathcal{T}(G^\star) = \mathcal{M}(G) \cup \{\mathbf{0}\}$. ■

Remark 7. *The proof of Theorem 8 provides an intuitive winning strategy for* k-pile Nim *without the mention of the concept 'Nim-sum'. Could, possibly, an averaged intelligent 5 year old child learn to use this strategy? Shift the pile with the least number of tokens towards you, say an inch, and take the same number of tokens from any of the other piles and put them next to the pile you had just 'shifted'. Continue in this same way with the remaining piles, until there is at most one pile of tokens left in the old place, 'an inch further away'. The correct winning move, provided there is one, is to remove these tokens. We consider this strategy as only a little less intuitive than the rules themselves. At least it does not require any binary arithmetics.*

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n	a_n	fib.repr.	b_n	fib.repr.	n	a_n	fib.repr.	b_n	fib.repr.
1	1	1	1	1	41	33	1010101	33	1010101
2	3	100	3	100	42	35	10000001	35	10000001
3	3	100	4	101	43	37	10000100	43	10010001
4	4	101	4	101	44	38	10000101	38	10000101
5	6	1001	6	1001	45	40	10001001	40	10001001
6	8	10000	8	10000	46	40	10001001	43	10010001
7	8	10000	9	10001	47	42	10010000	51	10100101
8	8	10000	12	10101	48	42	10010000	56	100000001
9	9	10001	9	10001	49	43	10010001	43	10010001
10	9	10001	12	10101	50	43	10010001	46	10010101
11	11	10100	11	10100	51	45	10010100	48	10100001
12	11	10100	12	10101	52	45	10010100	51	10100101
13	12	10101	12	10101	53	45	10010100	55	100000000
14	14	100001	14	100001	54	46	10010101	46	10010101
15	16	100100	17	100101	55	48	10100001	48	10100001
16	17	100101	17	100101	56	48	10100001	50	10100100
17	19	101001	19	101001	57	48	10100001	51	10100101
18	19	101001	21	1000000	58	48	10100001	55	100000000
19	21	1000000	21	1000000	59	50	10100100	59	100000101
20	21	1000000	25	1000101	60	51	10100101	51	10100101
21	21	1000000	30	1010001	61	53	10101001	55	100000000
22	21	1000000	33	1010101	62	53	10101001	56	100000001
23	22	1000001	22	1000001	63	53	10101001	59	100000101
24	22	1000001	25	1000101	64	53	10101001	63	100010000
25	22	1000001	33	1010101	65	55	100000000	67	100010101
26	24	1000100	25	1000101	66	55	100000000	72	100100101
27	24	1000100	27	1001001	67	55	100000000	77	101000001
28	24	1000100	33	1010101	68	55	100000000	80	101000101
29	25	1000101	25	1000101	69	55	100000000	85	101010001
30	25	1000101	33	1010101	70	55	100000000	88	101010101
31	27	1001001	27	1001001	71	56	100000001	56	100000001
32	27	1001001	33	1010101	72	56	100000001	59	100000101
33	29	1010000	33	1010101	73	56	100000001	67	100010101
34	29	1010000	35	10000001	74	56	100000001	88	101010101
35	30	1010001	30	1010001	75	58	100000100	64	100010001
36	30	1010001	33	1010101	76	58	100000100	67	100010101
37	32	1010100	32	1010100	77	58	100000100	69	100100001
38	32	1010100	33	1010101	78	58	100000100	88	101010101
39	32	1010100	35	10000001	79	59	100000101	59	100000101
40	32	1010100	42	10010000	80	59	100000101	67	100010101

TABLE 1. The terminal P-positions of $W^\star$ are of the form $\{0, x\}, x \in \mathbb{N}_0$. The first few P-positions which are not terminal are given here as $\{a_n, b_n\}$. We have included the coding of these positions in the Fibonacci/Zeckendorf numeration system.