THE $\star$-OPERATOR AND INVARIANT SUBTRACTION GAMES

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Abstract. An *invariant subtraction game* is a 2-player impartial game defined by a set of invariant moves (k-tuples of non-negative integers) $\mathcal{M}$. Given a position (another k-tuple) $\boldsymbol{x} = (x_1, \ldots, x_k)$, each option is of the form $(x_1 - m_1, \ldots, x_k - m_k)$, where $\boldsymbol{m} = (m_1, \ldots, m_k) \in \mathcal{M}$ (and where $x_i - m_i \geq 0$, for all i). Two players alternate in moving and the player who moves last wins. The set of non-zero P-positions of the game $\mathcal{M}$ defines the moves in the dual game $\mathcal{M}^\star$. For example, in the game of (2-pile Nim)* a move consists in removing the same positive number of tokens from both piles. Our main results concern a double application of $\star$, the operation $\mathcal{M} \to (\mathcal{M}^\star)^\star$. We establish a fundamental 'convergence' result for this operation. Then, we give necessary and sufficient conditions for the relation $\mathcal{M} = (\mathcal{M}^\star)^\star$ to hold. We show that it holds for example with $\mathcal{M} = k$-pile Nim.

1. Introduction and terminology

An *invariant subtraction game* [DR, LHF] is a two-player *impartial combinatorial game* (see [BCG] for a background on such games) defined on a set of *positions* represented as k-tuples $\boldsymbol{x} = (x_1, \ldots, x_k)$, where $k \in \mathbb{N} = \{1, 2, \ldots\}$ and $x_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The move options are determined by a set, $\mathcal{M} \subset \mathbb{N}_0^k \setminus \{\boldsymbol{0}\}$, of *invariant moves*. Each *option*, from a given position $\boldsymbol{x} = (x_1, \ldots, x_k)$, is of the form

$$\boldsymbol{x} \ominus \boldsymbol{m} = (x_1 - m_1, \ldots, x_k - m_k) \succeq \boldsymbol{0},$$

where $\boldsymbol{m} = (m_1 \ldots, m_k) \in \mathcal{M}$ (and $x_i - m_i \geq 0$ for all i). The players alternate in moving and the player who moves last wins. Clearly, this setting excludes the possibility of a draw game, but it includes many classical "take-away" games played on a finite number of tokens, e.g. Nim [B], Wythoff Nim [W], the (one-pile) subtraction games in [BCG].

Remark 1. *Our setting is very similar to the "take-away" games in* [G]*. However, since nowadays the term "take-away" often includes the possibility of a certain form of "move dependence"* [S, Z] *which we are not considering here, we prefer to use the terminology introduced in* [DR]*. Also, we differ from* [G] *in the definition of the ending condition of a game. Golomb's unique winning condition is a move to* $\boldsymbol{0}$*, so that in his setting many games are draw. (He also allows for the possibility of the vector* $\boldsymbol{0}$ *as a move.)*

Given an invariant subtraction game $\mathcal{M}$, we call a position N if the player about to move (the next player) wins. Otherwise it is P (the previous player

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wins). Hence, a position is P if and only if each option is N. A position $\boldsymbol{x}$ is *terminal* if $\mathbf{0} \preceq \boldsymbol{y} \preceq \boldsymbol{x}$ implies $\boldsymbol{y} \notin \mathcal{M}$. Hence, each terminal position is P. Altogether this gives that the sets of N- and P-positions are recursively defined. We denote these sets by $\mathcal{N}(\mathcal{M})$ and $\mathcal{P}(\mathcal{M})$ respectively.

Suppose that $X \subseteq \mathbb{N}_0^k$. Then, we denote by

$$X' = X \setminus \{\mathbf{0}\}.$$

Let $\mathcal{M}$ be an invariant subtraction game. Then the *dual game* of $\mathcal{M}$ is defined by

$$\mathcal{M}^\star = \mathcal{P}(\mathcal{M})'$$

and $\mathcal{M}$ is *reflexive* if

$$\mathcal{M} = \mathcal{P}(\mathcal{M}^\star)'.$$

(Thus we have departed somewhat from the terminology used in [LHF].) As before, we denote by $(\mathcal{M}^\star)^\star = \mathcal{M}^{\star\star}$.

A sequence of invariant subtraction games $(\mathcal{M}_i)_{i\in\mathbb{N}_0}$ *converges* if, for all $\boldsymbol{x} \in \mathbb{N}_0^k$, there is an $n_0 = n_0(\boldsymbol{x}) \in \mathbb{N}_0$ such that, for all $n \geq n_0$, for all $\boldsymbol{y} \preceq \boldsymbol{x}$, $\boldsymbol{y} \in \mathcal{M}_n$ if and only if $\boldsymbol{y} \in \mathcal{M}_{n_0}$. If $(\mathcal{M}_i)_{i\in\mathbb{N}_0}$ converges, then we can define the unique 'limit-game'

$$\lim_{i\in\mathbb{N}_0} \mathcal{M}_i. \tag{1}$$

For $i \in \mathbb{N}$, let $\mathcal{M}^i$ denote the game $(\mathcal{M}^{i-1})^\star$ and where $\mathcal{M} = \mathcal{M}^0$ is an invariant subtraction game.

Let us state our two main results, proved in Section 2 and 3 respectively.

Theorem 1. *Let $\mathcal{M}$ denote an invariant subtraction game. Then the sequence $(\mathcal{M}^{2i})_{i\in\mathbb{N}_0}$ converges.*

Let $X \subseteq \mathbb{N}_0^k$. Then we denote by $\mathcal{D}(X) = \{\boldsymbol{x} - \boldsymbol{y} \succ \mathbf{0} \mid \boldsymbol{x}, \boldsymbol{y} \in X\}$.

Theorem 2. *Let $\mathcal{M}$ denote an invariant subtraction game. Then the following items are equivalent,*

(a) *$\mathcal{M}$ is reflexive,*
(b) *$\mathcal{M} = \lim_{i\in\mathbb{N}_0} \mathcal{M}^{2i}$, for some invariant subtraction game $\mathcal{M}^0$,*
(c) *$\mathcal{D}(\mathcal{M}) \subset \mathcal{N}(\mathcal{M})$.*

In Figure 1 we demonstrate a simple application of Theorem 2 (c). In Figure 2 we show an example of a game which has a very simple structure, but for which we do not know whether reflexivity holds for any game resulting from a finite number of recursive applications of $\star$. (Due to computer simulations there appears to be many such games.) In Section 3 we study a consequence of Theorem 2, which relates to the type of question studied in [DR, LHF]. We give a partial resolution of the problem: Given a set $S \subset \mathbb{N}_0^k$, is there a game $\mathcal{M}$ such that $\mathcal{P}(\mathcal{M}) = S$?

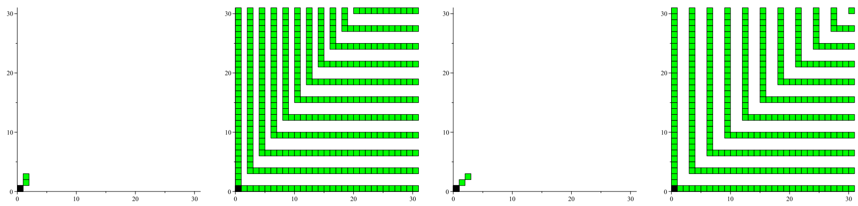

FIGURE 1. The figures illustrate three recursive applications of $\star$ on $\mathcal{M} = \{(1,1),(1,2)\}$ (for positions with coordinates less than 20). By Theorem 2 (c), $\mathcal{M}$ is not reflexive since $(1,2) \ominus (1,1) = (0,1) \in \mathcal{P}(\mathcal{M})$. Neither is the dual, $\mathcal{M}^\star$, since $(1,0)$ and $(3,2)$ are moves, but $(3,2) \ominus (1,0) = (2,2) \in \mathcal{P}(\mathcal{M}^\star)$. On the other hand $\mathcal{M}^{\star\star} = \{(1,1)(2,2)\}$ is reflexive, since $(2,2) \ominus (1,1) = (1,1) \in \mathcal{M}^{\star\star} \subset \mathcal{N}(\mathcal{M}^{\star\star})$. Hence $\mathcal{M}^n$ is reflexive for all $n \geq 2$.

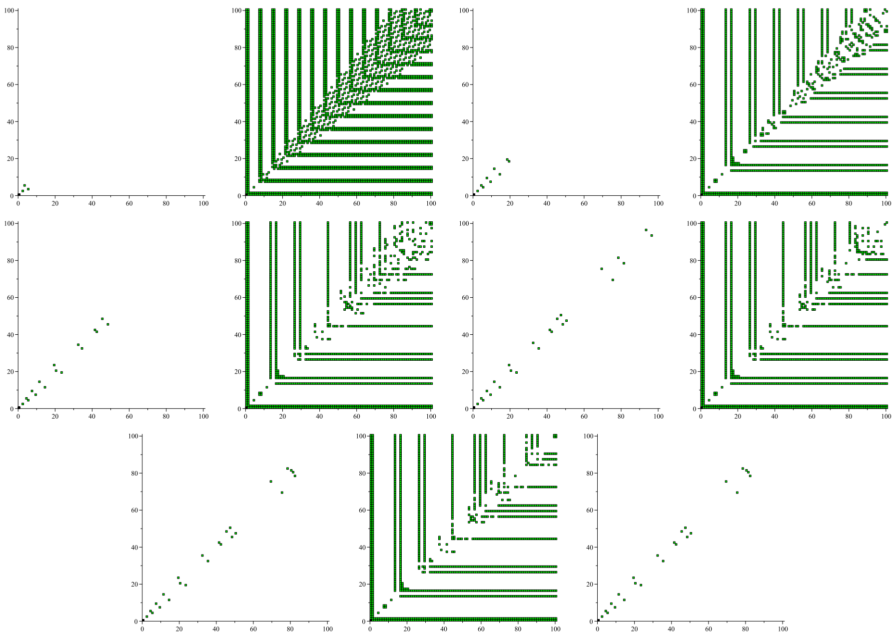

FIGURE 2. The figures illustrate 10 recursive applications of $\star$ on $\mathcal{M} = \{(2,2),(3,5),(5,3)\}$ (for positions with coordinates less than 100). Notice that $(3,5) \ominus (2,2) = (1,3) \in \mathcal{P}(\mathcal{M})$, so that by Theorem 2 (c), $\mathcal{M}$ is not reflexive (as is also clear by the figures). However, due to these experimental results, $\mathcal{M}^n \cap \{(i,j) \mid i,j \in \{0,1,\ldots,100\}$ is identical for $n = 8$ and $n = 10$ and hence, for all even $n \geq 8$ (and similarly for all odd $n \geq 9$). Of course, by Theorem 1, we get that $\lim \mathcal{M}^i$ exists. However, we do not know whether there exists an $n \geq 8$ such that $\mathcal{M}^n = \lim \mathcal{M}^i$ (see also Question 2 on page 9).

2. Convergence

Let us begin by proving Theorem 1. We omit a proof of the first lemma (see also [LHF]).

Lemma 1 ([LHF])**.** *Let $\mathcal{M}$ denote an invariant subtraction game. Then*
- (a) $\mathcal{P}(\mathcal{M}) \cap \mathcal{M} = \emptyset$,
- (b) $\mathcal{M}^\star \cap \mathcal{M} = \emptyset$, *and*
- (c) $\mathcal{P}(\mathcal{M}) \cap \mathcal{P}(\mathcal{M}^\star) = \emptyset$.

The next lemma concerns consequences of Lemma 1 for the $\star\star$-operator.

Lemma 2. *Let $\mathcal{M}$ denote an invariant subtraction game.*
- (a) *Suppose that $\boldsymbol{x} \in \mathcal{M} \setminus \mathcal{M}^{\star\star}$. Then $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}^\star) \setminus \mathcal{M}^\star$.*
- (b) *Suppose that $\boldsymbol{0} \prec \boldsymbol{x} \in \mathbb{N}_0^k$ is such that, for all $\boldsymbol{m} \prec \boldsymbol{x}$, $\boldsymbol{m} \in \mathcal{M}$ if and only if $\boldsymbol{m} \in \mathcal{M}^{\star\star}$. Then*

$$\boldsymbol{x} \notin \mathcal{M}^{\star\star} \setminus \mathcal{M}. \tag{2}$$

Proof. Assume that the hypothesis of item (a) holds. Then, since $\boldsymbol{x} \in \mathcal{M}$, by Lemma 1, $\boldsymbol{x} \notin \mathcal{P}(\mathcal{M})$, so that $\boldsymbol{x} \notin \mathcal{M}^\star$. Also, since $\boldsymbol{x} \notin \mathcal{M}^{\star\star}$, by definition of $\star$, we get that $\boldsymbol{x} \in \mathcal{N}(M^\star)$.

For (b), suppose that the negation of (2) holds, that is that $\boldsymbol{x} \in \mathcal{M}^{\star\star} \setminus \mathcal{M}$. Then

$$\boldsymbol{x} \in \mathcal{P}(\mathcal{M}^\star)', \tag{3}$$

which, by Lemma 1 (c), gives $\boldsymbol{x} \notin \mathcal{P}(\mathcal{M})$. Altogether, we get that $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}) \setminus \mathcal{M}$. Then, by definition of N, there is a move, say $\boldsymbol{m} \in \mathcal{M}$, with $\boldsymbol{m} \prec \boldsymbol{x}$, such that

$$\boldsymbol{y} = \boldsymbol{x} \ominus \boldsymbol{m} \in \mathcal{P}(\mathcal{M})' = \mathcal{M}^\star. \tag{4}$$

Then, by the assumption in the lemma, $\boldsymbol{m} \in \mathcal{M}^{\star\star} = \mathcal{P}(\mathcal{M}^\star)'$. By (3) and (4), this contradicts the definition of P in the game $\mathcal{M}^\star$. □

Proof (of Theorem 1). Let $\mathcal{M}$ denote an invariant subtraction game. Suppose that

$$\boldsymbol{x} \in \mathbb{N}_0^k \setminus \{\boldsymbol{0}\} \tag{5}$$

is such that, for all $\boldsymbol{y} \prec \boldsymbol{x}$,

$$\boldsymbol{y} \in \mathcal{M} \text{ if and only if } \boldsymbol{y} \in \mathcal{M}^{\star\star}. \tag{6}$$

Then clearly

$$\boldsymbol{y} \in \mathcal{P}(\mathcal{M}) \text{ if and only if } \boldsymbol{y} \in \mathcal{P}(\mathcal{M}^{\star\star}), \tag{7}$$

so that, by definition of $\star$,

$$\boldsymbol{y} \in \mathcal{M}^\star \text{ if and only if } \boldsymbol{y} \in \mathcal{M}^3.$$

Hence, a repeated application of $\star$ gives

$$\boldsymbol{y} \in \mathcal{M}^{2i} \text{ if and only if } \boldsymbol{y} \in \mathcal{M}^{2i+2}$$

and also

$$\boldsymbol{y} \in \mathcal{M}^{2i+1} \text{ if and only if } \boldsymbol{y} \in \mathcal{M}^{2i+3},$$

for all $i \in \mathbb{N}_0$. Suppose that $\boldsymbol{x}$ is of the form in (5). Then, by the definition of convergence, it suffices to demonstrate that the number, i, of applications of $\star$ on $\mathcal{M}$, so that

$$\boldsymbol{x} \in \mathcal{M}^{2i} \text{ if and only if } \boldsymbol{x} \in \mathcal{M}^{2i+2} \tag{8}$$

is bounded. Precisely, we will show that $i = 1$ suffices, which means that to satisfy (8), at most 2 iterations of $\star\star$ is needed for each position which satisfies the requirements of $\boldsymbol{x}$ in (6). Thus we show that, for any game $\mathcal{M}$ and any position $\boldsymbol{x}$, it suffices to take $n_0 = 2\prod_{i=1}^{k} x_i$ in the definition of convergence.

We have four cases,

(A) $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}) \cap \mathcal{N}(\mathcal{M}^{\star\star})$,

(B) $\boldsymbol{x} \in \mathcal{P}(\mathcal{M}) \cap \mathcal{P}(\mathcal{M}^{\star\star})$,

(C) $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}) \cap \mathcal{P}(\mathcal{M}^{\star\star})$ or

(D) $\boldsymbol{x} \in \mathcal{P}(\mathcal{M}) \cap \mathcal{N}(\mathcal{M}^{\star\star})$.

At first, notice that (B) together with Lemma 1 (a) implies $\boldsymbol{x} \notin \mathcal{M} \cup \mathcal{M}^{\star\star}$ (which gives $i = 0$ in (8)). Similarly, for case (D), by using Lemma 1 (a) twice, since $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})' = \mathcal{M}^{\star}$, we get $\boldsymbol{x} \notin \mathcal{M}$ and $\boldsymbol{x} \notin \mathcal{P}(\mathcal{M}^{\star})' = \mathcal{M}^{\star\star}$.

It remains to investigate case (A) and (C).

Case (A): By Lemma 2 (a), we may assume that $\boldsymbol{x} \in \mathcal{M} \backslash \mathcal{M}^{\star\star}$ (for otherwise we are done). By Lemma 2 (a), this gives that

$$\boldsymbol{x} \in \mathcal{N}(M^{\star}) \setminus \mathcal{M}^{\star}. \tag{9}$$

Hence, by definition of N, we get that there is a position $\boldsymbol{y} \in \mathcal{P}(\mathcal{M}^{\star})'$ such that

$$\boldsymbol{m} = \boldsymbol{x} \ominus \boldsymbol{y} \in \mathcal{M}^{\star}.$$

By (6) and (7) this implies that $\boldsymbol{y} \in \mathcal{P}(\mathcal{M}^3)$ and $\boldsymbol{m} \in \mathcal{M}^3$. If $\boldsymbol{x}$ were a move in $\mathcal{M}^4$ then, by definition of $\star$, $\boldsymbol{x} \in \mathcal{P}(\mathcal{M}^3)$. Altogether, this contradicts the definition of P. Hence, for this case, $\boldsymbol{x} \notin \mathcal{M}^4$, which suffices for convergence in this case.

Case (C): Since $\boldsymbol{x} \in \mathcal{N}(\mathcal{M})$, the definition of $\star$ gives $\boldsymbol{x} \notin \mathcal{M}^{\star}$. Hence, by $\boldsymbol{x} \in \mathcal{P}(\mathcal{M}^{\star\star})$ and Lemma 1 (c), we get that $\boldsymbol{x} \in \mathcal{N}(\mathcal{M}^{\star}) \setminus \mathcal{M}^{\star}$. As in the proof of (A), from (9) an onwards, this gives that $\boldsymbol{x} \notin \mathcal{M}^4$. Also, Lemma 1 (a), gives that $\boldsymbol{x} \notin \mathcal{M}^{\star\star}$, which proves convergence. □

3. Reflexivity

In this section we discuss criteria for reflexivity of a game. We begin by proving Theorem 2. Let us restate it.

Theorem 2. *Let $\mathcal{M}$ denote an invariant subtraction game. Then the following items are equivalent.*

(a) *$\mathcal{M}$ is reflexive,*
(b) *$\mathcal{M} = \lim_{i\in\mathbb{N}_0} \mathcal{M}^{2i}$, for some invariant subtraction game $\mathcal{M}^0$,*
(c) *$\mathcal{D}(\mathcal{M}) \subset \mathcal{N}(\mathcal{M})$.*

Proof. If $\mathcal{M} = \mathcal{M}^{\star\star}$ then $\mathcal{M}^{2i} = \mathcal{M}^{2i+2}$, for all $i \geq 0$, so that $\lim \mathcal{M}^{2i} = \mathcal{M}$. If $\mathcal{M} = \lim M^{2i}$ exists, then $\mathcal{M}^{\star\star} = (\lim \mathcal{M}^{2i})^{\star\star} = \lim \mathcal{M}^{2i} = \mathcal{M}$. Hence, it remains to prove that $\mathcal{M}$ is reflexive if and only if $D(\mathcal{M}) \subset \mathcal{N}(\mathcal{M})$.

"$\Rightarrow$": Suppose that $\mathcal{M}$ is reflexive. Then, we have to prove that $\mathcal{D}(\mathcal{M}) \subset \mathcal{N}(\mathcal{M})$. Suppose, on the contrary, that there are distinct $\boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{M}$ such that

$$\boldsymbol{m}_1 \ominus \boldsymbol{m}_2 = \boldsymbol{x} \in \mathcal{P}(\mathcal{M})'. \tag{10}$$

Then, by definition of $\star$,

$$\boldsymbol{x} \in \mathcal{M}^{\star}. \tag{11}$$

Also, by reflexivity, we get that $\{\boldsymbol{m}_1, \boldsymbol{m}_2\} \subset \mathcal{M}^{\star\star} = \mathcal{P}(\mathcal{M}^{\star})'$. But, by (10) and definition of P, this contradicts (11).

"$\Leftarrow$": Suppose that $\mathcal{D}(\mathcal{M}) \subset \mathcal{N}(\mathcal{M})$ but $\mathcal{M} \neq \mathcal{M}^{\star\star}$. Then there is some least $\boldsymbol{m} \in (\mathcal{M} \setminus \mathcal{M}^{\star\star}) \cup (\mathcal{M}^{\star\star} \setminus \mathcal{M})$, which, by Lemma 2 (b), gives $\boldsymbol{m} \in \mathcal{M} \setminus \mathcal{M}^{\star\star}$. As in the proof of Theorem 1, this gives $\boldsymbol{m} \in \mathcal{N}(\mathcal{M}^{\star}) \setminus \mathcal{M}^{\star}$. Then, by definition of N, there are $0 \prec \boldsymbol{x}, \boldsymbol{y} \prec \boldsymbol{m}$, with $\boldsymbol{x} \in \mathcal{M}^{\star}$ and $\boldsymbol{y} \in \mathcal{P}(\mathcal{M}^{\star})$, such that

$$\boldsymbol{m} \ominus \boldsymbol{x} = \boldsymbol{y}. \tag{12}$$

Then, by definition of $\star$, $\boldsymbol{y} \in \mathcal{M}^{\star\star}$ and so, by minimality of $\boldsymbol{m} \in \mathcal{M} \setminus \mathcal{M}^{\star\star}$, we must have $\boldsymbol{y} \in \mathcal{M}$. But, the definition of $\star$ also gives $\boldsymbol{x} \in \mathcal{P}(\mathcal{M})$, which, by the assumption $\mathcal{D}(\mathcal{M}) \subset \mathcal{N}(\mathcal{M})$, contradicts (12). □

By Theorem 2 (c), one never needs to compute $\mathcal{P}(\mathcal{M}^{\star})$ to understand the reflexivity properties of a game $\mathcal{M}$. Even more is true for many games $\mathcal{M}$. Sometimes a very incomplete understanding of the winning strategy $\mathcal{P}(\mathcal{M})$ suffices. Namely, to disprove reflexivity it suffices to find a single move which 'connects' any two P-positions. On the other hand, to prove reflexivity, it suffices to find some subset $X \subset \mathcal{N}(\mathcal{M})$ such that $\mathcal{D}(\mathcal{M}) \subseteq X$ holds.

In particular, if we take $X = \mathcal{M}$ we obtain very simple reflexivity properties. Namely, whenever $\mathcal{D}(\mathcal{M}) \subset \mathcal{M} \subseteq \mathcal{N}(\mathcal{M})$, the game $\mathcal{M}$ is 'trivially' reflexive, that is, no knowledge of the winning strategy of $\mathcal{M}$ is required to establish reflexivity.

Let $X \subset \mathbb{N}_0^k$. Then the set X is

- *subtractive* if, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, with $\boldsymbol{x} \prec \boldsymbol{y}$, $\boldsymbol{y} \ominus \boldsymbol{x} \in X$.
- a *lower ideal* if, for all $\boldsymbol{y} \in X$, $\boldsymbol{x} \prec \boldsymbol{y}$ implies $\boldsymbol{x} \in X$. (Hence the set of terminal P-positions of a given game constitutes a lower ideal.)
- an *anti-chain*, if all distinct pairs $\boldsymbol{x}, \boldsymbol{y} \in X$ are unrelated, that is $\boldsymbol{x} \preceq \boldsymbol{y}$ implies $\boldsymbol{x} = \boldsymbol{y}$.

We have the following corollary of Theorem 2 (see also Figure 3 for an application of (a)).

Corollary 1. *The game $\mathcal{M}$ is reflexive if, regarded as a set,*

(a) *$\mathcal{M}$ is subtractive,*
(b) *$\mathcal{M}$ is a lower ideal,*
(c) $\mathcal{M} = \{(x, 0, \ldots, 0), (0, x, 0, \ldots, 0), \ldots, (0, \ldots, 0, x) \in \mathbb{N}_0^k \mid x \in \mathbb{N}\}$, *that is $\mathcal{M}$ represents the classical game of k-pile Nim* [B]*,*
(d) *$\mathcal{M}$ is an anti-chain, or*
(e) $\mathcal{M} = \{\boldsymbol{m}\}$*, that is $\mathcal{M}$ consists of a single move.*

Proof. For (a), notice that, by Theorem 2,

$$\mathcal{D}(\mathcal{M}) = \{\boldsymbol{m}_1 \ominus \boldsymbol{m}_2 \succ \boldsymbol{0} \mid \boldsymbol{m}_1, \boldsymbol{m}_2 \in \mathcal{M}\} \subseteq \mathcal{M} \subseteq \mathcal{N}(\mathcal{M}),$$

which gives the claim. Then, the inclusions of families of games $\{\mathcal{M}_e\} \subset \{\mathcal{M}_d\} \subset \{\mathcal{M}_a\}$ and $\{\mathcal{M}_c\} \subset \{\mathcal{M}_b\} \subset \{\mathcal{M}_a\}$ prove the corollary, where $\mathcal{M}_a$ denotes the game given by a set $\mathcal{M}$ as in item (a) etc. □

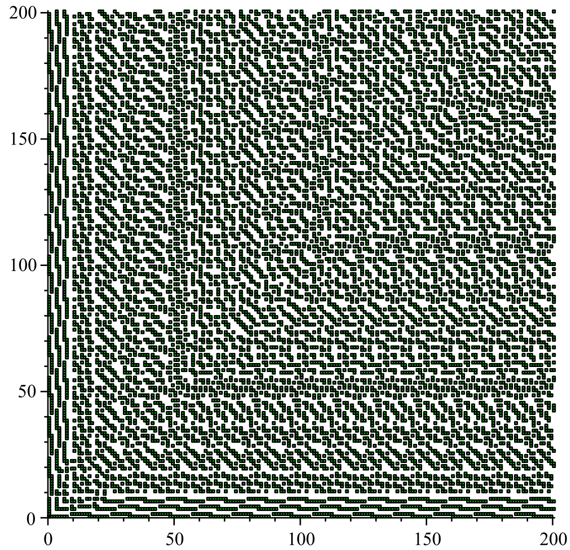

Figure 3. The game $\{(1,1),(2,2),(0,8),(8,0)\}$ is subtractive and hence, by Corollary 1, reflexive. The figure represents its first few P-positions. (In Figure 1, $\mathcal{M}^{\star\star}$ is subtractive, but $\mathcal{M}$ is not.) Hence the dual is also reflexive (but not subtractive). In spite of the simplicity of the game rules, its set of P-positions seem to have a very complex structure (in the sense of [F2]). By the experimental result in this figure, it seems to be 'a-periodic' in general, but 'asymptotically periodic' for each fixed x-coordinate (or y-coordinate), but we do not understand these patterns. See also the final section for a comment regarding undecidability of games with a finite number of moves.

Due to this discussion, we believe that there are many interesting applications of Theorem 2. Let us begin with two.

3.1. **A consequence of reflexivity.** Given a set $S \subset \mathbb{N}_0^k$, is there an invariant subtraction game $\mathcal{M}$ such that $\mathcal{P}(\mathcal{M}) = S$? This type of question was introduced in [DR], together with a challenging conjecture on a family of sets $S \subset \mathbb{N}_0^2$ defined by a certain class of increasing sequences of positive integers. (The conjecture was resolved in [LHF].) As a consequence of Theorem 2 (and Corollary 2), we are able to shed some new light on this type of question for general sets S.

Corollary 2. *Let $S \subset \mathbb{N}_0^k \setminus \{\mathbf{0}\}$, $k \in \mathbb{N}$, and suppose that S is reflexive, so that, by Theorem 2,*

$$\mathcal{D}(S) \subseteq \mathcal{N}(S). \tag{13}$$

Then, there is a game $\mathcal{M}$ satisfying

$$\mathcal{P}(\mathcal{M})' = S. \tag{14}$$

For the other direction, (13) holds if and only if there is a game $\mathcal{M}$ which satisfies both (14) and

$$\mathcal{M} = \mathcal{P}(S)'. \tag{15}$$

Proof. Suppose that (13) holds. Then, by Theorem 2, the game S is reflexive, so that $S = S^{\star\star}$. Take $\mathcal{M} = S^{\star}$. Then, the definition of $\star$ gives the first claim. (Because $\mathcal{P}(\mathcal{M})' = \mathcal{P}(S^{\star})' = S^{\star\star} = S$.)

For the second part, suppose that there is no game $\mathcal{M}$ such that (15) holds (here S is regarded as a game). Then, for all $\mathcal{M}$ such that (14) hold, we have that

$$S^{\star\star} = \mathcal{P}(\mathcal{P}(S)')' \neq \mathcal{P}(\mathcal{M})' = S,$$

and so, by Theorem 2, since reflexivity of S does not hold neither does (13).

If, on the other hand, (15) and (14) hold for one and the same game $\mathcal{M}$, then the definition of $\star$ gives that (13) holds. □

It is easy to find a (non-reflexive) set S which does not satisfy (14) for any $\mathcal{M}$ (see also [DR, LHF] and [G, Theorem 3.2] for a related result).

Example 1. *Let $S = \{(1,1),(1,2)\}$ (see also Figure 1). Then $\mathcal{D}(S) = \{(0,1)\} \subset \{(0,x) \mid x \in \mathbb{N}_0\} \subset \mathcal{P}(S)$ so that reflexivity of S does not hold. Also, for our choice of S, there cannot be any game $\mathcal{M}$ satisfying (14). Indeed, by the definition of N, $(0,1)$ has to be a move, which contradicts the definition of P since $(1,2) \ominus (1,1) = (0,1)$.*

Neither is it hard to find a set S which satisfies (14) but not (15), although strictly more than two (candidate) P-positions are needed.

Example 2. *Suppose that $S = \{(0,1),(1,0),(1,1),(3,3)\}$. Then the first part of the corollary does not give any information on whether there is a game $\mathcal{M}$ such that (14) holds. Namely we have that $(2,2) \in \mathcal{D}(S) \cap \mathcal{P}(S)$, which contradicts (13) (and thus reflexivity of S). However, by inspection one finds that $S \subset \mathcal{P}(\mathcal{Q})$ for $\mathcal{Q} = \{(0,2),(2,0),(1,2),(2,1)\}$. Then, in spite of the observation that S is not reflexive, this gives the existence of a game $\mathcal{M}$ satisfying (14). (For example take $\mathcal{M} = \mathcal{Q} \cup \{(x,y),(y,x) \mid x \geq 4\}$.)*

3.2. **Decidability and reflexivity.** A very simple configuration of moves, e.g. Figure 3, can have a very 'complex' set of P-positions (dual game). In fact, suppose the invariant subtraction game $\mathcal{M} \subset \mathbb{N}_0^k$ has finite cardinality. Then we wonder whether it is algorithmically decidable if a given k-tuple ($\succ \mathbf{0}$) appears as a difference of any two P-positions of $\mathcal{M}$. (In [LW] we have proved undecidability in a related sense for a similar class of invariant games.) However, by Theorem 2, since $\mathcal{D}(\mathcal{M})$ is finite if $\mathcal{M}$ is, the question whether a certain finite configuration of moves is reflexive or not must be decidable. Hence we get another corollary of Theorem 2

Corollary 3. *Suppose that the number of moves in the invariant subtraction game $\mathcal{M}$ is finite. Then it takes at most a finite computation to decide whether $\mathcal{M}$ is reflexive or not.*

4. Discussion

In this paper we have presented some general territory of invariant subtraction games and the $\star$-operator. The issues of convergence of the $\star\star$-operator have been completely resolved, but we have not found any explicit formula for a 'non-trivial limit-game' as in (1). By 'trivial limit-game' we here mean a game H which satisfies $H = \mathcal{M}^{2n} = \lim \mathcal{M}^{2i}$ for some $n \in \mathbb{N}$ and some game $\mathcal{M}$.

Problem 1. *Give an explicit formula for a non-trivial limit game. That is, give an explicit (tractable* [F2]*) formula for its set of moves (without the mention of a limit of a sequence of games).*

Our next question is a continuation of the examples in Section 3.

Question 1. *Examples 1 and 2 suggest a classification of non-reflexive sets $S \subset \mathbb{N}_0^k$, that is, by Theorem 2, sets for which there exists a pair $\boldsymbol{x}, \boldsymbol{y} \in S'$ such that $\boldsymbol{x} \ominus \boldsymbol{y} \in \mathcal{P}(S')$. The first class should contain those sets S for which there exist a game $\mathcal{M}$ such that $\mathcal{P}(\mathcal{M})' = S$ and the second, those for which there is no such game. Suppose there exists a pair $\boldsymbol{x}, \boldsymbol{y} \in S$ such that the only possible 'candidate move' from $\boldsymbol{m} = \boldsymbol{x} \ominus \boldsymbol{y}$ to another position in $S \cup \{\mathbf{0}\}$ is to $\mathbf{0}$. Then, we are in Example 1 and so in the second class. On the other hand, Example 2 gives an example when there is no such pair $\boldsymbol{x}, \boldsymbol{y}$. But suppose that the positions $(2,3)$ and $(3,2)$ are included to the set S in Example 2. Then, neither the move $(2,2)$ nor the moves $(1,2)$ and $(2,1)$ may be included to the candidate set $\mathcal{M}$, and hence S would have belonged to the second class. Is there an explicit and exhaustive classification which settles the type of question suggested by Example 1 and 2?*

In Figure 2 we gave an example of a non-reflexive game with a non-reflexive dual, but where the dual of the dual is reflexive. In the example of the 'symmetric' game $\mathcal{M} = \{(2,2),(3,5),(5,3)\}$ from Figure 2 contains only three moves, but I was not able to determine whether there is an n such that $\mathcal{M}^n$ is reflexive or not. This discussion leads us to our final question.

Question 2. *Is there, for each $n \in \mathbb{N}$, a game $\mathcal{M}$ such that $\mathcal{M}^n$ is reflexive, but $\mathcal{M}^{n-1}$ is not?*

We do not know if the answer to Question 2 is positive for any $n \geq 3$.

References

[BCG] E. R. Berlekamp, J. H. Conway, R. K. Guy, *Winning ways*, **1-2** Academic Press, London (1982). Second edition, **1-4**. A. K. Peters, Wellesley/MA (2001/03/03/04).

[B] C. L. Bouton, Nim, A Game with a Complete Mathematical Theory *The Annals of Mathematics*, 2nd Ser., Vol. 3, No. 1/4. (1901 - 1902), pp. 35-39.

[DR] E. Duchêne and M. Rigo, Invariant Games, *Theoret. Comp. Sci.*, Vol. 411, 34-36 (2010), pp. 3169-3180

[F2] A. S. Fraenkel, Complexity, appeal and challenges of combinatorial games, Proc. of Dagstuhl Seminar "Algorithmic Combinatorial Game Theory", *Theoret. Comp. Sci.* **313** (2004) 393-415, special issue on Algorithmic Combinatorial Game Theory.

[G] S. W. Golomb, A mathematical investigation of games of "take-away". *J. Combinatorial Theory* **1** (1966) 443—458.

[LHF] U. Larsson, P. Hegarty, A. S. Fraenkel, Invariant and dual subtraction games resolving the Duchêne-Rigo Conjecture, *Theoret. Comp. Sci.* Vol. 412, 8-10 (2011) pp. 729-735.

[LW] U. Larsson, J. Wästlund, From heaps of matches to undecidability of games, preprint.

[S] A. J. Schwenk, "Take-Away Games", *Fibonacci Quart.* **8** (1970), 225-234.

[Z] Michael Zieve, Take-Away Games, *Games of No Chance, MSRI Publications*, **29**, (1996) pp. 351361

[W] W.A. Wythoff, A modification of the game of Nim, *Nieuw Arch. Wisk.* **7** (1907) 199-202.