MAHARAJA NIM

WYTHOFF'S QUEEN MEETS THE KNIGHT

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Abstract. New combinatorial games are introduced, of which the most pertinent is Maharaja Nim. The rules extend those of the well-known impartial game of Wythoff Nim in which two players take turn in moving a single Queen of Chess on a large board, attempting to be the first to put her in the lower left corner. Here, in addition to the classical rules a player may also move the Queen as the Knight of Chess moves. We prove that the P-positions of Maharaja Nim are close to the ones of Wythoff Nim, namely they are within a bounded distance to the lines with slope $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}-1}{2}$ respectively. For a close relative to Maharaja Nim (where the Knight's jumps are of the form $(2,3)$ and $(3,2)$ rather than $(1,2)$ and $(2,1)$) we also demonstrate polynomial time complexity to the decision problem of whether a given position is P.

1. Maharaja Nim

We introduce a 2-player combinatorial game called *Maharaja Nim*, an extension of the well-known game of Wythoff Nim [Wy]. (The name 'Maharaja' is taken from a variation of Chess, 'The Maharaja and the Sepoys', [Fa].) Both these games are impartial, that is, the set of options are the same regardless of whose turn it is. For a background on impartial games see [BCG].

Place a Queen (of Chess) on a given position of a large Chess board, with the position in the lower left corner labeled $(0,0)$. In the game of Wythoff Nim, here denoted by W, the two players move the Queen alternately as it moves in Chess, but with the restriction that, by moving, no coordinate increases, see Figure 1. Only non-negative coordinates are allowed so that the first player to reach the position $(0,0)$ wins.

In Maharaja Nim, denoted by M, the rules are as in Wythoff Nim, except that the Queen is exchanged for a 'Maharaja', a piece which may move both as the Queen and the Knight of Chess, again, provided by moving no coordinate increases. See Figure 1.

We say that a position is P if the second player to move has a winning strategy, otherwise N. Also, let $\mathcal{P}_{\mathrm{M}}$ and $\mathcal{P}_{\mathrm{W}}$ denote the set of P-positions of Maharaja Nim and Wythoff Nim respectively.

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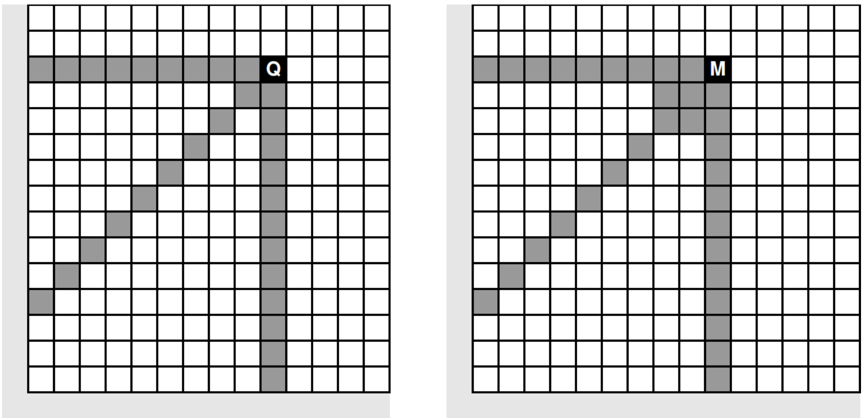

FIGURE 1. The moves of Wythoff Nim and Maharaja Nim respectively.

We let $\mathbb{N}$ denote the positive integers and $\mathbb{N}_0$ the non-negative integers. Let

$$\phi = \frac{1+\sqrt{5}}{2}$$

denote the golden ratio. The well-known winning strategy of Wythoff Nim [Wy] is

$$\mathcal{P}_{\mathrm{W}} = \{(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor), (\lfloor \phi^2 n \rfloor, \lfloor \phi n \rfloor) \mid n \in \mathbb{N}_0\}. \tag{1}$$

From this it follows that there is precisely one P-position of Wythoff Nim in each row and each column of the board (see also [Be]).

The purpose of this paper is to explore the P-positions of Maharaja Nim and some related games. Clearly $(0,0)$ is P. Another trivial observation is that, since the rules of game are symmetric, if (x,y) is P then (y,x) is P. It is also easy to see that there is at most one P-position in each row and each column (corresponding to the Rook-type moves). But, in fact, the same assertion as for Wythoff Nim holds:

Proposition 1.1. *There is precisely one P-position of Maharaja Nim in each row and each column.*

Proof. Since all Nim-type moves are allowed in Maharaja Nim, there is at most one P-position in each row and column of $\mathbb{N}_0 \times \mathbb{N}_0$. This implies that there are at most k P-positions strictly to the left of the k^{th} column (row). Each such P-position is an option for at most three N-positions in column (row) k. This implies that there is a least position in column (row) k which has only N-positions as options. By definition this position is P and so, since k is an arbitrary index, the result follows. □

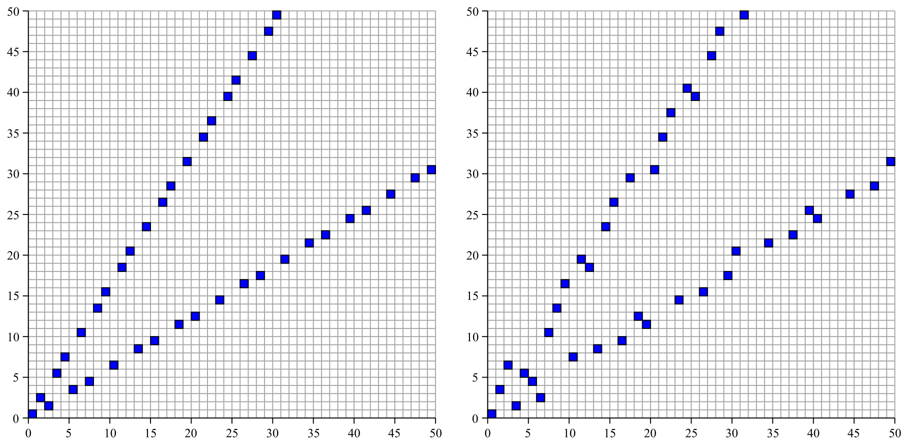

FIGURE 2. The initial P-positions of Wythoff Nim and Maharaja Nim respectively.

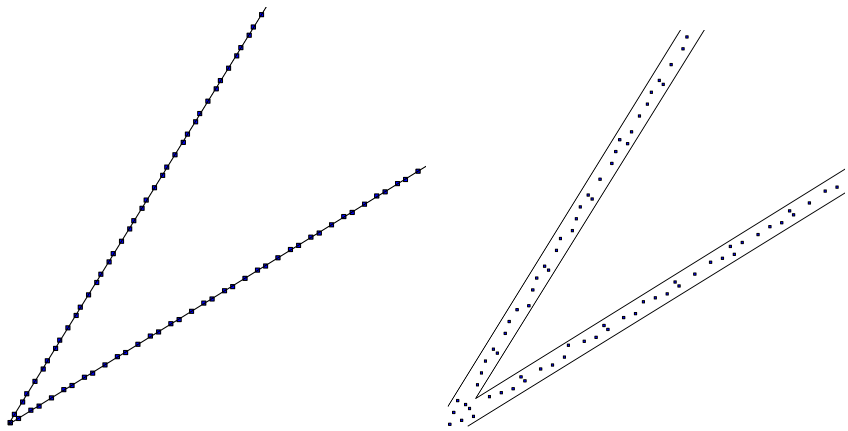

FIGURE 3. To the left, the P-positions of Wythoff Nim lie 'on' the lines ϕx and $\phi^{-1}x$, $x \geq 0$. The figure to the right illustrates a main result of this paper, that the P-positions of Maharaja Nim are bounded below and above by the 'bands', $\phi x + O(1)$ and $\phi^{-1}x + O(1)$

Another claim holds for both Wythoff Nim and Maharaja Nim: There is *at most* one P-position on each 'diagonal' of the form

$$\{\{x, x + C\} \mid x \in \mathbb{N}_0\}, C \in \mathbb{N}_0, \tag{2}$$

(corresponding to the Bishop-type moves). But (1) readily gives that, for Wythoff Nim, there is *precisely* one P-position on each such diagonal. Even more is true: If

$$\mathcal{P}_{\mathrm{W}} = \{(a_i, b_i), (b_i, a_i)\}, \tag{3}$$

with (a_i) increasing and for all i, $a_i \leq b_i$, then for all n,

$$\{0, 1, \ldots, n\} = \{b_i - a_i \mid i \in \{0, 1, \ldots, n\}\}. \tag{4}$$

As we will see in Section 2, a somewhat weaker, but crucial, property holds also for Maharaja Nim, but let us now state our main result (see also Figure 3). We let $O(1)$ denote bounded functions on $\mathbb{N}_0$.

Theorem 1.2 (Main Theorem)**.** *Each P-position of Maharaja Nim lies on one of the 'bands'* $\phi n + O(1)$ *or* $\phi^{-1} n + O(1)$*, that is, if* $(x, y) \in \mathcal{P}_M$*, with* $y \geq x$*, then* $y - \phi x$ *is* $O(1)$*.*

We give the proof of this result in Section 2. It is quite satisfactory in one sense, but for the two gamesters trying to figure out how to quickly find safe positions, it does not quite suffice. The following question is left open.

Question 1. *Does Maharaja Nim's* decision problem*, to determine whether a given position* (x, y)*, with input length* $\log(xy)$*, is P, has polynomial time complexity in* $\log(xy)$*?*

In Section 5 we provide an affirmative answer of this question for a close relative of Maharaja Nim, namely the extension of Wythoff Nim where moves of type $(2, 3)$ and $(3, 2)$ are adjoined (but not $(1, 2)$ or $(2, 1)$). This result builds upon an analog result, of 'approximately linear' P-positions, as that for Maharaja Nim in Theorem 1.2. See also the very interesting paper [FP], which was the soul inspiration for some results in this paper, although its methods do not seem to encompass the complexity of Maharaja Nim.

1.1. **Complementary sequences and a central lemma.** We say that two sequences of positive integers are *complementary* if each positive integer is contained in precisely one of these sequences. In [FP] the authors proved the following result.

Proposition 1.3 (Fraenkel, Peled)**.** *Suppose* x *and* y *are complementary and increasing sequences of positive integers. Suppose further that there is a positive real constant,* δ*, such that, for all* n*,*

$$y_n - x_n = \delta n + O(1). \tag{5}$$

Then there are constants, $1 < \alpha < 2 < \beta$*, such that, for all* n*,*

$$x_n - \alpha n = O(1) \tag{6}$$

and

$$y_n - \beta n = O(1). \tag{7}$$

As they have remarked (see also [HL]), by simple density estimates one may decide the constants α and β as functions of δ. Namely, notice that (5) and (6) together imply

$$\beta = \alpha + \delta \tag{8}$$

and, by complementarity, we must have

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1. \tag{9}$$

(Thus α and β are algebraic numbers if and only if δ is.) By this we get the relation

$$\delta(1-\alpha) + \alpha = (\alpha - 1)\alpha, \tag{10}$$

which will turn out to be useful in what will come next, namely we have found a short proof of an extension of their theorem—an extension which is easier to adapt to the circumstances of Maharaja Nim. Let us explain.

If we denote

$$\mathcal{P}_{\mathrm{M}} = \{(a_n, b_n), (b_n, a_n) \mid n \in \mathbb{N}_0\}, \tag{11}$$

with (a_n) increasing and for all $n, b_n \geq a_n$, then, for all n, b_n is uniquely defined by the rules of M. At this point, one might want to observe that, if the b-sequence would have been increasing (by Figure 2 it is not) then Theorem 1.2 would follow from Proposition 1.3 if one could only establish the following claim: $b_n - a_n - n$ is $O(1)$. Namely in (10) $\delta = 1$ gives $\alpha = \phi$ in Proposition 1.3. Now, luckily, it turns out that Proposition 1.3 holds without the condition that the y-sequence is increasing, namely (5) together with an increasing x-sequence suffices.

Lemma 1.4 (Central Lemma)**.** *Suppose x and y are complementary sequences of positive integers with x increasing. Suppose further that there is a positive real constant, δ, such that, for all n,*

$$y_n - x_n = \delta n + O(1). \tag{12}$$

Then there are constants, $1 < \alpha < 2 < \beta$, such that, for all n,

$$x_n - \alpha n = O(1) \tag{13}$$

and

$$y_n - \beta n = O(1). \tag{14}$$

Proof. We begin by demonstrating that, for all $n \in \mathbb{N}$,

$$x_{n+1} = x_n + O(1), \tag{15}$$

and

$$y_{n+1} = y_n + O(1). \tag{16}$$

By (12), for all $k, n \in \mathbb{N}$ we have that

$$\begin{aligned} y_{n+k} - y_n &= x_{n+k} + \delta(n+k) - x_n - \delta n + O(1), \\ &= x_{n+k} - x_n + \delta k + O(1). \end{aligned} \tag{17}$$

Since $x_{n+k} - x_n \geq k$ and $\delta > 0$ this means that, on the one hand, for all $k, n \in \mathbb{N}$,

$$y_{n+k} \geq y_n - C, \tag{18}$$

for some positive integer constant C (which may depend on δ). On the other hand, with C as in (18), (17) readily gives that there is a constant $\kappa = \kappa(C)$ such that, for all n,

$$y_{n+\kappa} - y_n \geq \kappa + 2C + 1. \tag{19}$$

On the one hand, there can be at most $\kappa - 1$ numbers from the y-sequence, with indexes strictly in-between n and $n + \kappa$, strictly in-between y_n and $y_{n+\kappa}$. On the other hand the inequality (18) gives that there can be at most C numbers from the y-sequence with index greater than $n+\kappa$ but less than $y_{n+\kappa}$. It also gives that there can be at most C numbers with index less than n but greater than y_n. Therefore, by complementarity and (19), there has to be a number from the x-sequence in every interval of length $\kappa + 2C + 1$. Thus the jumps in the x-sequence are bounded, which is (15). But then (16) follows from (12) and (15) since

$$\begin{aligned} y_{n+1} - y_n &= x_{n+1} + \delta(n+1) - x_n - \delta n + O(1) \\ &= x_{n+1} + \delta - x_n + O(1) \\ &= O(1). \end{aligned}$$

By (16) we may define m as a function of n with

$$x_n = y_m + O(1). \tag{20}$$

For example, we may take $m = m(n)$ the least number such that $x_n < y_m$. (With y increasing and $x_1 = 1$ this would imply, for all n, $x_n = n + m - 1$.) This has two consequences, of which the first one is

$$x_n = n + m + O(1). \tag{21}$$

This follows since the numbers $1, 2, \ldots, x_n$ are partitioned in n numbers from the x-sequence, and the rest, that is, by (18), $m + O(1)$ numbers from the y-sequence. Indeed, on the one hand, there can be at most C numbers from the y-sequence less than y_m but with an index greater than m. This gives $x_n \leq n + m + C$. On the other hand, to show $x_n \geq n + m - C$, notice that there can be at most C y-numbers with an index strictly less than m greater than x_n, for otherwise, by complementarity, there is an $0 < i < m$ such that $y_{m-i} - y_m > C$, contradicting (18).

The second consequence of (20) is that, by using (12),

$$x_n = x_m + \delta m + O(1). \tag{22}$$

If $\lim x_n/n$ and $\lim y_n/n$ exist then, clearly they must satisfy (8) and (9) with δ as in the lemma. Thus, using this definition of $\alpha = \alpha(\delta)$, for all n, denote

$$\Delta_n := x_n - \alpha n.$$

We want to use (21) and (22) to express Δ_n in terms of Δ_m.

Equation (22) expresses x_n in terms of x_m and m. Therefore, we wish to combine (21) and (22) to express n in terms of x_m and m, that is, we wish to eliminate x_n from (21). If we plug in the expression (22) for x_n in (21) and solve for n we get

$$n = x_m + (\delta - 1)m + O(1). \tag{23}$$

Combining (22) and (23) gives

$$\begin{aligned}\Delta_n &= x_m + \delta m - \alpha(x_m + (\delta - 1)m) + O(1)\\ &= (1-\alpha)x_m + (\delta(1-\alpha) + \alpha)m + O(1)\\ &= (1-\alpha)\Delta_m + O(1), \end{aligned} \tag{24}$$

where the last equality is by (10).

Notice that, by (22), for sufficiently large n we have that $m < n$. Hence we may use strong induction and by (24) conclude that Δ_n is $O(1)$ which is (13). Then (14) follows from (12). □

2. Perfect sectors, a dictionary and the proof of Theorem 1.2

This whole section is devoted to the proof of Theorem 1.2. We begin by proving that there is precisely one P-position of Maharaja Nim on each diagonal of the form in (2). Then we explain how the proof of this result leads to the second part of the theorem, the bounding of the P-positions within the desired 'bands' (Figure 3).

A position, say (x, y), is an *upper* position if it is strictly above the *main diagonal*, that is if $y > x$. Otherwise it is *lower*.

We call an 'upper perfect sector' or just a 'perfect sector' all positions strictly above some diagonal of the form in (2) and to the right of some column. Suppose that we have computed all P-positions in the columns $1, 2, \ldots, n-1$ and that, when we erase each upper position from which a player can move to an upper P-position (Figures 4 and 5), then the remaining upper positions to the right of and including column n constitute a *perfect sector* (Figure 5). Then we call column n *perfect*. It is easy to see that, if the n^{th} column is perfect then the property (4) holds for this particular n. On the other hand, the 'converse' to this holds if and only if we adjoin the condition that

$$b_n - a_n = n. \tag{25}$$

(As we will see in Section 5, this does not hold in general for relatives of Maharaja Nim). But, in fact, it is crucial to our approach that the first implication can be made stronger in the sense of (25).

Lemma 2.1. *Let $n \in \mathbb{N}_0$ and define (a_i) and (b_i) as in (11). Then*

$$\{0, 1, \ldots, n-1\} = \{b_i - a_i \mid 0 \le i < n\} \tag{26}$$

together with (25) hold if and only if the a_n^{th} column is perfect.

Proof. For the 'if' direction we are already done with (26). Notice that, $b_{n-1} = a_{n-1} + n - 1$ implies $a_n > a_{n-1} + 1$, for otherwise column a_n is not perfect. The position $(a_{n-1} + 1, b_{n-1} + 2)$ is erased, but belongs to the perfect sector. If, on the other hand, $b_{n-1} < a_{n-1} + n - 1$ then, by (26), the position $(a_{n-1} + 1, b_{n-1} + 2)$ has already been erased by some Bishop-type move to an upper P-position with a lower index. Hence, by definition of P, we may conclude that b_n takes the least possible value, namely $b_n = a_n + n$.

For the other direction, if both (25) and (26) hold, we use that $a_n = a_{n-1} + 1$ together with (25) imply that $b_{n-1} < a_{n-1} + n - 1$. Otherwise, if $a_n > a_{n-1} + 1$, then a Knight-type move from (a_n, b_n) to a P-position in column a_{n-1} (or lower) requires $b_n < a_n + n$, which is impossible. □

2.1. **Constructing Maharaja Nim's bit-string.** We study a bit-string, a sequence of '0's and '1's, where the i^{th} bit equals '0' if and only if there is an upper P-position of Maharaja Nim in column i. By Proposition 1.1, if there is no upper P-position in column i, there is a lower ditto (the i^{th} bit equals 1).

Suppose that column n is perfect. Then, by symmetry we know some lower P-positions in columns to the right of n. The next step is to erase each column in this perfect sector which has a lower P-position, a '1' in the bit-string (see Figure 6) and to, recursively in the non-erased part of the perfect sector, compute new upper P-positions. We do this until we reach the next perfect sector (for the moment assume that this will happen) at say column $n + m$, $m > 0$. Thus, using this notation, we may define a word of length m, containing the information of whether the P-position in column $i \in \{n, n+1, \dots, n+m-1\}$ is below or above the main diagonal.

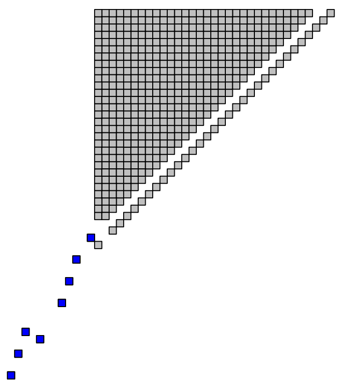

FIGURE 4. All upper positions from which a player can move to an upper P-position are erased. (The 'sector' continues above the figure.) However, the 'sector' is not perfect.

At this point we adjoin this *word* together with its unique *translate* to Maharaja Nim's *dictionary*. The translate is obtained accordingly: For each

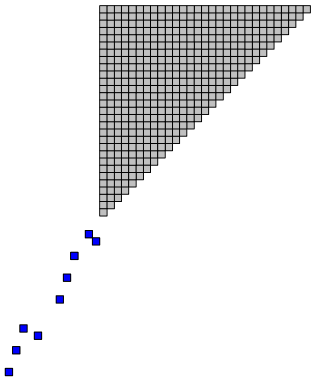

FIGURE 5. (Step 1) A perfect sector together with the corresponding initial P-positions.

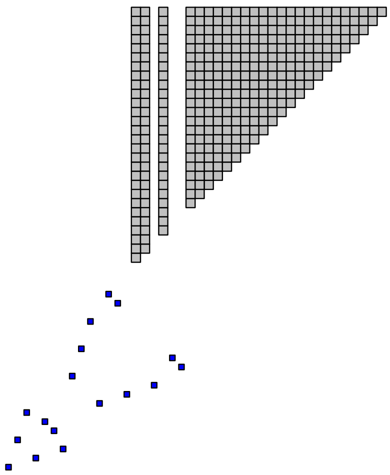

FIGURE 6. (Step 2) Each column in the perfect sector which corresponds to a lower P-position (a '1' in the bit-string) has been erased.

P-position in the columns n to $n+m-1$ define the i^{th} bit in the translation as a '1' if and only if row $k+i$ has an upper P-position and where k is the largest row index strictly below the perfect sector. See also Figure 7 and the next section for examples. Then the translate has length $m+l$, where l denotes the number of '0's in the word.

We then concatenate the translate at the end of the existing bit-string. In this way, provided a next perfect sector will be detected, the bit-string will always grow faster than we read from it. However, there is no immediate guarantee that we will be able to repeat the procedure—that the next word exists—or for that matter that the size of the dictionary will be finite, so that the process may be described by a finite system of words and translates.

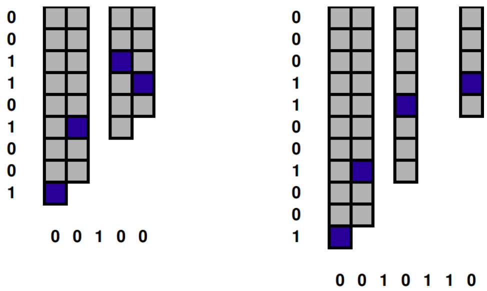

FIGURE 7. To the left, the unique P-positions of Maharaja Nim in the columns 8 to 12 are computed. The corresponding translation is $00100 \to 100101100$. To the right are the P-positions in the columns 14 to 20 together with the translate $0010110 \to 10010011000$. (Here we have omitted column 13 with its translation $1 \to 0$.) See also Figure 2 and Section 2.2.

But, in the coming, we aim to prove that, in fact, the next perfect sector will always (in the sense outlined above) be detected within a 'period' of at most 7 P-positions, that is '0's in the bit-string. As we will see, a complete dictionary needs only (between 9 and) 14 translations.

Let us describe a bit more in detail how the first part of Maharaja's bit-string is constructed.

2.2. **A detailed example.** Initially there is some interference which does not allow a recursive definition of words and translates, see Figure 2. The first perfect sector beyond the origin is attained when the four first P-positions strictly above the main diagonal has been computed. This happens to the right of column 8. To the right of column 12 a new perfect sector is detected. Thus the first word (left hand side entry) in the dictionary will be '00100', corresponding to the P-positions $(8, 13)$, $(9, 16)$, $(10, 7)$, $(11, 19)$ and $(12, 18)$. (Here there is no interference since the y-coordinate of the first P-position is greater than the x-coordinate of the last P-position.) Let us verify that this word translates to '100101100'. Notice that the first '1'-bit means that the P-position $(8, 13)$ is to the left of the main diagonal—by symmetry this corresponds to a lower P-position in column 13. The second bit is '0'. This means that the next upper P-position is in column 14. Then, by rules of game, it has to be at least in row 16, which indeed will be attained, so that the next P-position will be $(9, 16)$. By the rules of game, the rows 14 and 15 cannot have P-positions to the left of the main diagonal, so that a prefix is '1001'. Continuing up to the last P-position of this translate, $(12, 18)$, extends the prefix to '1001011'. The next upper P-position will be in at least row 22 since the least unused diagonal is $22 - 13 = 9$. After this a new perfect sector will start. This gives the two last '0's in the translate, '100101100', which may now be concatenated at the end of the first part of the bit-string, '00100', so that the new bit-string becomes '00100$\underline{1}$00101100'.

In column 13 there is a lower P-position (corresponding to the 6^{th} bit in the string), which gives a new perfect sector by default, that is, the next left hand side word is '1'. This corresponds to that the first column in a perfect sector is erased and we get a new perfect sector without adding any upper P-position. By the property of a perfect sector, there can be no P-position to the left of the main diagonal in row 22, so the translate of the word '1' must be '0'. A concatenation of this '0' at the end of the existing string gives '001001$\underline{0}$01011000'. As we continue to read from the '0' in the seventh position it turns out that, this time, we need to read '0010110' (Figure 7 to the right) to obtain a new perfect sector and also that this word translates to '10010011000'. Again, concatenating this translate at the end of the existing string gives '00100100101100$\underline{0}$010010011000', and so on.

2.3. **Maharaja Nim's dictionary.** The dictionary of M is

$$1 \to 0 \tag{27}$$
$$01 \to 100 \tag{28}$$
$$00100 \to 100101100 \tag{29}$$
$$00110 \to 10010100 \tag{30}$$
$$000100 \to 10010110100 \tag{31}$$
$$001110 \to 100100100 \tag{32}$$
$$0010110 \to 10010011000 \tag{33}$$
$$00000100 \to 100101100111000 \tag{34}$$
$$000010010 \to 1001001111000100 \tag{35}$$
$$0000000 \to 10010110110100 \tag{36}$$
$$0010100 \to 100100110100 \tag{37}$$
$$0011110 \to 1001000100 \tag{38}$$
$$00000010 \to 100101101100100 \tag{39}$$
$$00001000 \to 100100111100100. \tag{40}$$

By computer simulations we have verified that each one of the words (27) to (35) does appear in Maharaja Nim's bit-string. By our method of proof, we have found no way to exclude the latter five, but a guess is that they do not appear. At least they do not appear among the first 20000 bits of the bit-string. The following result gives the first part of the theorem.

Lemma 2.2 (Completeness Lemma)**.** *When we read from Maharaja Nim's bit-string each prefix is contained in our extended dictionary of (left hand side) words of Maharaja Nim.*

Proof. Let us present a list in lexicographic order of all words in our extended dictionary together with the words we need to exclude:

$$\begin{aligned}
0000000 &\to 10010110110100\\
00000010 &\to 100101101100100\\
00000011 &\text{ 'to exclude' (a)}\\
00000100 &\to 100101100111000\\
00000101 &\text{ 'to exclude' (b)}\\
0000011 &\text{ 'to exclude (c)'}\\
00001000 &\to 100100111100100\\
000010010 &\to 1001001111000100\\
000010011 &\text{ 'to exclude' (d)}\\
0000101 &\text{ 'to exclude' (e)}\\
000011 &\text{ 'to exclude' (f)}\\
000100 &\to 10010110100\\
000101 &\text{ 'to exclude' (g)}\\
00011 &\text{ 'to exclude' (h)}\\
00100 &\to 100101100\\
0010100 &\to 100100110100\\
0010101 &\text{ 'to exclude' (i)}\\
0010110 &\to 10010011000\\
0010111 &\text{ 'to exclude' (j)}\\
00110 &\to 10010100\\
001110 &\to 100100100\\
0011110 &\to 1001000100\\
0011111 &\text{ 'to exclude' (k)}\\
01 &\to 100\\
1 &\to 0
\end{aligned}$$

This list is 'complete' in the sense that any bit-string has precisely one of the words on the left hand side as a prefix. This motivates why it suffices to exclude the words 'to exclude'. For example (a) needs to be excluded since the only word in our list beginning with '0000001' ends with a '0'. Neither can we translate words beginning with '000001's and ending with '01' or '1'. This motivates why we need to exclude (b) and (c). All left hand side words in our dictionary beginning with 4 '0's continues with 100, which motivates that (e) and (f) need to be excluded, and so on. We move on to verify that the strings (a) to (k) are not contained in the bit-string.

No translate can contain more than three consecutive '0's. To get a longer string one has to finish off one translate and start a new. The only translate which starts with '0' is '0'. Thus, when a sequence of four or more '0's is interrupted it means that a new translate has begun. But all translates that begin with a '1' begins with '100'. Thus, a sequence of four or more '0's cannot be followed by '11' or '101'. This gives that the exclusion of the words (a),(b), (c), (e) and (f) is correct.

Clearly, the string '100' in (d) has to be the prefix of some translate. Since the next two bits are '11', by the dictionary, this translate has to be '100'. But then the next translate has the prefix '11', which is impossible.

For the exclusion of (g) and (h) notice that the only strings of three consecutive '0's that exist within a translate is either at the end or is followed by the string '100'. Therefore, a string of three '0's cannot be followed by '11' or '101'.

For (i), notice that the sub-string '101010' is not contained in any translate. If it were, it needed to be either at the beginning of a translate, which is impossible (since all of them except '0' begin with '100') or be split between two. The latter is impossible since all translates except '0' ends with '00'. In analogy to this, also (j) must be excluded and similarly for (k) since no translate contains 5 consecutive '1's and all translates ends in a '0', but starts with either '0' or '10'. □

Since the left hand side words have at most 7 '0's we adjoin at most 6 P-positions in a sequence with $b_n - a_n$ distinct from n. Namely, by Lemma 2.1, when we start a new perfect sector we know that the next P-position will satisfy $b_n - a_n = n$. The number of bits in a translate is bounded (by 16) so that b_n can never deviate more than a bounded number of positions from $a_n + n$. Hence, by Proposition 1.1, the conditions of Lemma 1.4 are satisfied with the a-sequence as x, the b-sequence as y and $\delta = 1$. Thus, $b_n - a_n - n$ is $O(n)$ (as discussed in the paragraph before Lemma 1.4) this concludes the proof of Theorem 1.2.

3. Dictionary processes and undecidability

Let us briefly discuss a problem related to the method used in this paper. Given a dictionary (of binary words and translations) and a starting string, will the translation process of the bit-string 'terminate' or not?

More precisely, let us assume that we have a finite list of words $A = \{A_1, A_2, \ldots, A_m\}$ with translates $B_1, B_2, \ldots, B_m$ respectively, each word being a string of '0's and '1's, and where we assume that none of the words in A is a prefix of another. The latter is not a necessary condition, as we will explore further in Section 5. Namely, as the read head reads from the bit-string, a natural generalization of a prefix free dictionary is to translate precisely the longest word containing a certain prefix.

Take any string S as a starting string (for example A_1 but it could be an arbitrary string, not necessarily in the list). A 'read head' '_' starts to read S from left to the right and as soon as it finds a string A_i in A it stops, sends a signal to a printer at the other end which concatenates the translation B_i at the end of S. Then the read head continues to read from where it ended until it finds the next word in A, its translation being concatenated at the end, and so on.

If the read head gets to the end of the string without finding a word in the list A, the process stops with the current string as 'output'. Otherwise, the process continues and gives as output an infinite string.

It follows from E. Post's tag productions [Mi, Po] that it is algorithmically undecidable whether our 'dictionary processes' stop or not. We omit a further analysis of these type of questions since it would divert us from the main subject of the paper, analyzes of P-positions of combinatorial games.

4. Approximate linearity, converging dictionaries and polynomial time complexity

There are infinitely many relatives to Maharaja Nim of the form 'adjoin a finite set of moves to Wythoff Nim'. It is easy to see that the conclusion of Proposition 1.1 holds for all these games. For any given such generalization, is it possible to determine the greatest departure from n for $b_n - a_n$? For example see the games in Figure 8 and 9. Even simpler, is it decidable, whether there is a P-position above some straight line? More precisely:

Question 2. *Given the moves of Wythoff Nim together with some finite list of moves, that is ordered pairs of integers (in Maharaja Nim the list is* $\{(1,2),(2,1)\}$*) and a linear inequality in two variables x and y, is it decidable whether there is a P-position in the game which satisfies the inequality?*

On the one hand it is not even clear if a 'generalized Maharaja Nim' has a finite dictionary in the sense of Section 2. On the other hand the solution of a similar game may or may not depend on the possible outcome of a dictionary process as in Section 2. In fact, in Section 5 we prove that a related dictionary process is successful in giving a polynomial time algorithm for the decision problem of whether a certain position is P. Therefore, let us look into some questions regarding some close relatives of Maharaja Nim.

To begin with, one might want to pay special attention to the family of extensions of Wythoff Nim, where the adjoined moves are of the form (k,l) and (l,k), $k,l \in \mathbb{N}$, $k < l$. We call a game in this family (k,l)-Maharaja Nim, (k,l)M. (Although another interesting path of exploration is indicated in Figure 9 and its discussion.) The P-positions are distinct from those of Wythoff Nim, see [La], if and only if (k,l) is a so-called 'Wythoff pair' or a 'dual Wythoff pair', that is of the form $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor)$ or $(\lceil \phi n \rceil, \lceil \phi^2 n \rceil)$, $n \in \mathbb{N}$. Thus, in Maharaja Nim we take the first Wythoff pair $(1,2) = (\lfloor \phi \rfloor, \lfloor \phi^2 \rfloor)$, whereas in the next section we study $(2,3)$-Maharaja

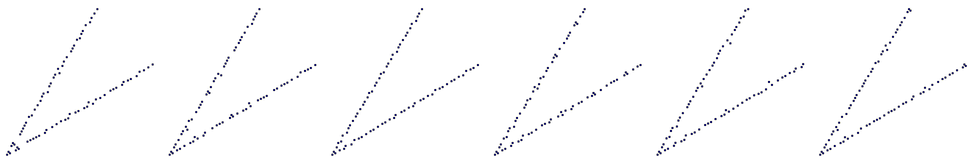

FIGURE 8. The initial P-positions (the coordinates are less than 100) of (k,l)M for $(k,l) = (3,5),(4,6),(4,7),(5,8),(6,10)$ and $(7,11)$ respectively. In support of Conjecture 4.1, the ratios of the respective coordinates seem to closely approximate ϕ or $1/\phi$. (For $(2,3)$M, see Section 5.)

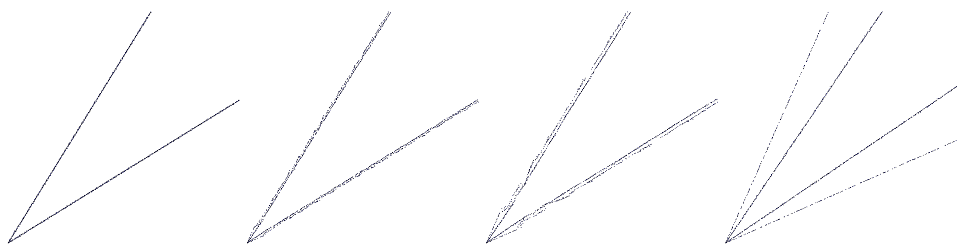

FIGURE 9. The initial P-positions (the coordinates are less than 1500) of four extensions of Maharaja Nim where the adjoined moves are $\{(t,2t),(2t,t)\}$ where $t \in \{1,2,\ldots,10\}$, $\{1,2,\ldots,50\}$, $\{1,2,\ldots,100\}$ and $\mathbb{N}$ respectively. That is the three first games have a finite number of adjoined moves to Wythoff Nim but the last one has infinitely many. Notice the seemingly emerging 'bounded split' of the (upper) P-positions in the middle two figures, the ratio of the coordinates still seem to be within a bounded distance of ϕ, but in the last figure, where an infinite number of moves are adjoined the convergence to ϕ is destroyed, a fact which is proved in [La], and a (unbounded) 'split' seems to be established (conjectured in [La]).

Nim, that is we let (k,l) take the values of the first dual Wythoff pair $(2,3) = (\lceil\phi\rceil, \lceil\phi^2\rceil)$.

Conjecture 4.1. *Let $k,l \in \mathbb{N}$, $k < l$. Then each upper P-position (x,y) of $(k,l)M$ satisfies $y = \phi x + O(1)$.*

Does this conjecture hold for any game of the form 'a finite number of adjoined moves to Wythoff Nim'?

Suppose that a given game (k,l)M has a finite (non-terminating) dictionary (as for Maharaja Nim) thus, hypothetically, providing an affirmative answer to Conjecture 4.1. Suppose further that the dictionary *converges*, that is, given an arbitrary string-position, we can, within the distance of a

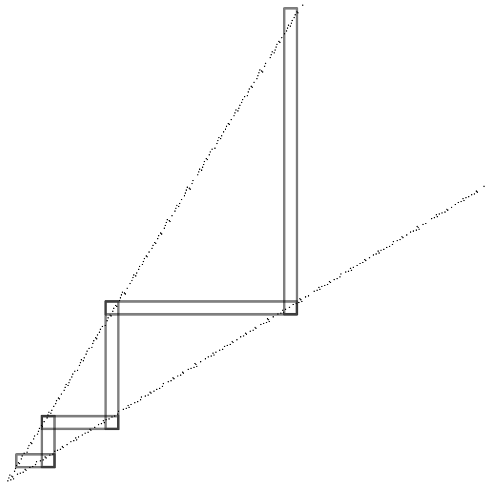

Figure 10. A 'telescope' with 'focus' $O(1)$ and 'reflectors' along the lines ϕn and n/ϕ attempts to determine the value (P or N) of some position, (x, y) at the top of the picture. As we demonstrate in Section 5 the method is successful for $(2, 3)$-Maharaja Nim. (It gives the correct value for all extensions of Wythoff Nim with a finite non-terminating converging dictionary). The focus is kept sufficiently wide (a constant) to provide correct translations in each step. The number of steps is linear in $\log(xy)$.

bounded number of bits, precisely determine when a new word starts. For this particular game, let us sketch a polynomial time algorithm which determines whether a given position (x, y) (with $\frac{y}{x}$ approximately ϕ) is P, see also Figure 10. Suppose that we have computed an initial (sufficiently large) sequence of the bit-string. We sketch the steps of the decision problem of (k, l)M are as follows:

- 'Back track' (x, y) via orthogonal 'reflections' along the lines ϕn and n/ϕ. Here we do not need to use our dictionary, only to put marks at the precise locations of our reflecting points on the lines ϕn and n/ϕ. That is, we get a finite sequence of pairs of the form $(x, \phi x), (x, x/\phi), (x/\phi^2, x/\phi) \ldots, (x/\phi^p, x/\phi^{p-1})$, some $p \in \mathbb{N}$.
- When we have back tracked as far as to our initial bit-string, the 'forward' translations can begin. Suppose that we know that the dictionary converges within q (which is supposed to be much less than x and y) bits and that the maximal length of a translate is $c \leq q$ bits.
- Then it suffices to translate $< \phi q$ bits in each step. If the first left hand side word begins with, say the bit $\lfloor x/\phi^p \rfloor - \phi q \leq b_1 \leq \lfloor x/\phi^p \rfloor - q$ we may translate it and be assured to find another left hand side word beginning at a bit $\lfloor x/\phi^{p-1} \rfloor - \phi q \leq b_2 \leq \lfloor x/\phi^{p-1} \rfloor - q$ and so on.

For the final computation of the value of (x, y) it suffices to, given the left hand side word which contains x, compute the P-positions in some area of size less than $c \times c$ squares. (Alternatively, given a short dictionary, the list of P-positions corresponding to each word may be computed beforehand.)

- This procedure takes p steps where ϕ^p is proportional to $x + y$.

5. The close relative (2, 3)-Maharaja Nim has a polynomial time complexity

The game (2, 3)-Maharaja Nim, (2, 3)M, is as Maharaja Nim except that, for this game, the Knight's jumps are of the form (2, 3) and (3, 2) (and not (1, 2) and (2, 1)). In this section we let $(a_1, b_1), (a_2, b_2), \ldots$ denote the upper P-positions of (2, 3)M, where (a_i) is increasing. As we have remarked in Section 4 an analog of Proposition 1.1 holds for (2, 3)M. Hence (a_i) and (b_i) are complementary.

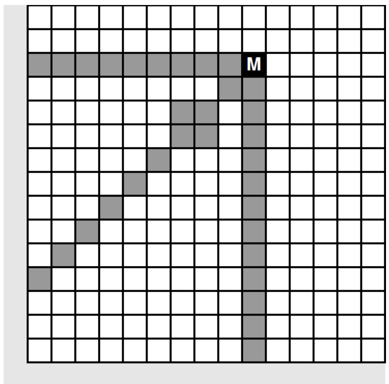

Figure 11. The moves of (2, 3)-Maharaja Nim.

Since Lemma 2.1 does not hold for (2, 3)M, for the analysis of this game we use a relaxation of the approach in Section 2. As we saw at the end of that section, the crucial property for approximate linearity to hold is that the dictionary promised a sufficiently frequent reappearance of property (25). Hence, for a new left hand side word to be translated it is not necessary that we require a perfect sector to be detected. It turns out that the other condition in Lemma 2.1 suffices for our purposes, let us explain.

Suppose that the initial P-positions up to column a_n has been coded in a unique (2, 3)M bit-string, where as before, a '1' ('0') in th i^{th} position denotes a lower (upper) P-position in column i. That is the read head is about to read the a_n^{th} bit in the string. As in Section 2, by symmetry of P-positions, a finite number of bits follow to the right of the read head's current position. Then a (new) left hand side word $X \neq 1$ (the word '1' is translated to '0') is included to the 'dictionary' if and only if the following two criterion are satisfied. Each one of the numbers $0, 1, \ldots, n - 1$ is represented as

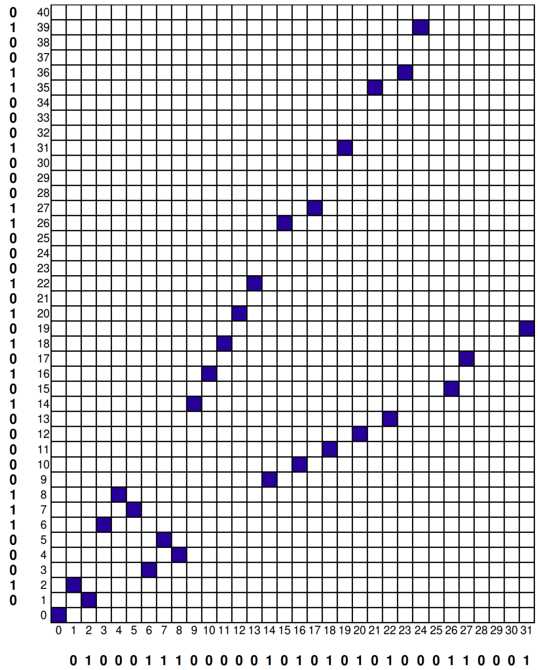

FIGURE 12. The initial P-positions of $(2,3)$-Maharaja Nim together with its initial bit-string.

the positive difference of the coordinates of precisely one of the first $n-1$ upper P-positions *and* the number n represents the corresponding difference of the n^{th} upper P-position. That is we require that (25) and (26) hold simultaneously.

As usual, the translation of X is computed and concatenated at the end of the bit-string. The next left hand side word begins by the a_n^{th} column.

5.1. $(2,3)$**-Maharaja Nim's dictionary process.** By adapting the above rules for the left hand side words of $(2,3)$-Maharaja Nim, we may define the following very short (but non-prefix-free) 'dictionary' of $(2,3)$M:

$$0 \to 10 \tag{41}$$

$$1 \to 0 \tag{42}$$

$$01000 \to 100011100 \tag{43}$$

$$01010 \to 10001100. \tag{44}$$

Since the bit '0' is a prefix of the words '01000' and '01010' we need some external rule to decide which translation to use in the construction of the bit-string. The rule we have in mind is simple. Suppose that the next bit detected by the read head is '0'. Then the translation is as in (41), except if the next four bits are either '1000' or '1010'. For these cases the translations are as in (43) and (44) respectively. Notice that, by these translation rules some advantage has appeared (which were not present in Maharaja Nim's dictionary process) namely, by (41) and (42), *any* bit-string has a translation, and therefore it cannot terminate. Before proving that this dictionary is correct, let us provide some initial example.

Column-wise, the first non-terminal P-positions of $(2,3)$M are $(1,2)$, $(2,1), (3,6), (4,8), (5,7), (6,3), (7,5)$ and $(8,4)$. These P-positions correspond to the bit-string '01000111' on the x-axis and '0100011100000' on the y-axis, see Figure 12. That is, the last five '0's are adjoined so that the first word to be translated starts in position $(9,14)$, which corresponds to the first 'free' diagonal (of the form (2)) in the 9^{th} column. The initial 'interference' between rows and columns ends here so, to begin with the read heads position is '01000111$\underline{0}$0000'. Thus, the first four translations are of the type (41) which produce the bit-string '010001110000$\underline{0}$10101010'. Then a type (44) translation follows which produces '01000111000001010$\underline{1}$01010001100', and so on.

It is easy to see that, given a perfect sector to the right of the column of the read head's position, each translation in $(2,3)$M's dictionary is correct. However, since a priori there is no guarantee that a new translation starts at a perfect sector, we need to (be able to) exclude certain combinations of translations, thus preventing any $(2,3)$-type move from 'P to P' a so-called 'short-circuit' of two P-positions. The translations (41) and (43) could potentially interfere with a succeeding translation but (42) and (44) cannot. Precisely, if the word '0' were followed by a '1' and then any of the words beginning with '0', or if the word '01000' were followed by a '0', then the translation rules would be wrong, because of a $(2,3)$-type 'short-circuit'. These are all cases 'to exclude'. Let us begin to rule out the latter case.

Claim: The left hand side word '01000' is always succeeded by the left hand side word '1'.

Proof: Suppose, on the contrary, that the read head reads the left hand side word '01000' followed by a '0'. This string, '010000', must have been translated from the words 'x','0','1','1','1' (in this order and where x is the word in either (41), (43) or (44)). But the string '00111' only appears as a translation in (43). Further, the string '01000111' is forced since '11000111' cannot appear, but it cannot be that the read head detects the first five bits '01000' as the word in (43), since then we would have started with the string '100011100000' which is not what we wanted. It follows that either of the strings

(45) 010001000111,

(46) 0101000111, or

(47) 010101000111

must have been read, with a new left hand side word starting from the first '0'. But then, for the case (45), this gives the translate '1000111000101010000', which forces that a left hand side word starts after '111', that is '01010' will be detected as a word and so the word '01000' will not be read, which contradicts our initial assumption. For the latter cases (46) and (47), we get the translates '100011001010000' and '1000110001010100000' respectively, both which may be treated in analogy to the first case, but here it is forced that new left hand side words start after the '11' strings respectively. □

The first case concerns any sequence of left hand side words beginning with '0', '1' and then some sequence '0xy', where x and y represent two bits. It is immediate by the translation rules that we may exclude the cases where xy represents '00' or '10'. Namely, for these two cases, by the 'prefix-rule' of choosing the longest left hand side word in the dictionary, we would rather have used one of the translations in (43) or (44). Also, the case where xy is '11' may be excluded since the string '01011' does not appear in any combination of translates. Thus, it only remains to analyze the case where the two bits are '01'. That is, we want to exclude the pattern '01001'. By looking at the translations it is obvious that the string '1001' must have been translated from the left hand side words '0', '1' and then a word beginning with a '0'. This means that precisely the pattern which we want to exclude has appeared in a previous translation (and thereby short-circuiting two P-positions in columns strictly to the left of the current position). Thus (using Figure 12 as a base case) strong induction resolves this case.

5.2. **Polynomiality.** We have proved that the dictionary in (41) to (44) is correct and thereby also that the P-positions of $(2, 3)$-Maharaja Nim lie within a bounded distance of either the 'line' ϕn or $\phi^{-1} n$. Next, we will demonstrate that this dictionary gives a polynomial strategy, as outlined in Section 4. For this, it suffices to prove that, given an arbitrary position in the infinite bit-string, by a search within a bounded number of bits we can determine which of the four given translations is correct.

If the read head reads the pattern '11' then, by the left hand side words in the dictionary and in particular (42), we can conclude that a new word starts by the first '1'. Hence, from some bit and onwards, assume that only combinations of the translates in (41) and (42) are detected. By analyzing the translations in the dictionary one can see that at most five consecutive '0's can appear. Therefore, we may assume that the read head reads the pattern '01', which, by the above 'first case' proof, leads to either '01000' or '01010'. Both these strings are detected as words, unless the preceding pattern ends with '0100', '01' or '0101'. Hence one needs to investigate the following six ambiguous strings:

(a) 010001000,
(b) 0101000,
(c) 010101000,

(d) 010001010,
(e) 0101010,
(f) 010101010.

Since we have excluded the translates (43) and (44), the pattern '10001000' in (a) must have been translated from the string '011011', which is nonsense. So (a) cannot appear.

The string in (b) must have been translated from '0011' which, by (43) and (44) and since all translates end with a '0', implies that the three preceding bits must have been '010'. Hence, we can extend the pattern to be translated to '0100011'. It is given that the prefix '01000' of this string cannot be detected as a left hand side word. Therefore, the translation of '0100011' must be '10010101000' which has the prefix

$$\text{'1001'}, \tag{48}$$

but, by the left hand side words in the dictionary, any string containing (48) must converge between the two '0's (here a new word must start as '01010' followed by '1', '0', '0',...). Notice that (c) is a suffix of this string, so it may also be included in the argument. Also, by (48), (d) must be preceded by the pattern '01', but then again, we may analyze (d) as (b).

We are left with the strings (e) and (f). Since a repetition of more than five consecutive patterns '01' implies that more than five consecutive 0s has been translated, which is impossible, we may assume that the repetitions of '01' in (f) has been preceded by either of the patterns '10' or '00' ('11' is already ruled out). Again, the first case leads to (48). Notice that (e) can also be included in this argument. For the second case, notice that any string beginning with '00001' converges after the three first '0's, that is a new word must begin with '01', so it suffices to study the string '1000101010', which (since the pattern '11' is excluded) has been treated already in (d).

We have proved that, given an arbitrary position in the bit-string, at most a bounded number of preceding bits need to be searched in order to find the correct translation. By Section 4 this convergence gives a polynomial strategy of $(2, 3)$-Maharaja Nim.

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