A GENERALIZED DIAGONAL WYTHOFF NIM

URBAN LARSSON

ABSTRACT. The P-positions of the well-known 2-pile take-away game of Wythoff Nim lie on two 'beams' of slope $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}-1}{2}$ respectively. We study extensions to this game where a player may also remove simultaneously pt tokens from either of the piles and qt from the other, where $p < q$ are given positive integers and where t ranges over the positive integers. We prove that for certain pairs (p, q) the P-positions are identical to those of Wythoff Nim, but for $(p, q) = (1, 2)$ they do not even lie on two beams. In fact, we conjecture a classification of all pairs (p, q) for which each of the two beams of P-positions of Wythoff Nim, in the new game, 'splits' into two distinct new beams of P-positions.

1. INTRODUCTION

We study generalizations of the 2-player impartial take away games of 2-pile Nim [Bou02] and Wythoff Nim, [Wyt07, Fra82, FrOz98, HeLa06, Lar09, Lar, LaWä]. A background on impartial (take away) games can be found in for example [BCG82, Con76]. We use some standard terminology for such games without draw positions. A position is a previous-player win, a *P-position*, if none of its options is, otherwise it is a next-player win, an *N-position*. We follow the conventions of *normal play*, that is, the player who moves last wins. Thus, given an impartial game, we get a recursive characterization of the set of all P-positions.

Let $\mathbb{N}$ denote the positive integers and $\mathbb{N}_0$ the non-negative integers. A legal move in 2-pile Nim is to remove an arbitrary number of tokens from precisely one of the piles, at least one token and at most the whole pile. This type of moves can be coded in the form $(0, t)$, $(t, 0)$, $t \in \mathbb{N}$ and it is easy to see that the P-positions of this game constitute the equal pile heights, coded in the form (x, x), $x \in \mathbb{N}_0$, [Bou02]. These positions may be regarded as one singular infinite North-East P-*beam*, with its source at the origin. In the game of Wythoff Nim a player may move as in Nim but also *diagonally*, (t, t), $t \in \mathbb{N}$. (Remove the same number of tokens from each pile but at most a whole pile.) For this game, geometrically, the singular Nim-beam of P-positions has 'split' into precisely two distinct beams (with sources at the

---

origin) one leaning towards 'North' and the other towards 'East'. Let

$$\phi = \frac{\sqrt{5}+1}{2}$$

denote the Golden ratio. It is well-known [Wyt07] that a position of this game is P if and only if it belongs to the set

$$\mathcal{P}(\text{WN}) = \{(\lfloor \phi x \rfloor, \lfloor \phi^2 x \rfloor), (\lfloor \phi^2 x \rfloor, \lfloor \phi x \rfloor) \mid x \in \mathbb{N}_0\}$$

so that the new P-beams have slopes ϕ and $1/\phi$ respectively. This geometrical observation has motivated us to ask the following intuitive question. Does this 'splitting behavior', going from Nim to Wythoff Nim, continue in some meaningful way if we adjoin, to the game of Wythoff Nim, some *generalized diagonal* moves of the form

$$(1) \qquad\qquad\qquad (pt, qt) \text{ and } (qt, pt),$$

where $p, q \in \mathbb{N}$ are fixed game parameters and where t ranges over $\mathbb{N}$, and then play the new game with both the old and the new moves? (That is, in addition to the rules of Wythoff Nim, a legal move is to remove simultaneously pt tokens from either of the piles and qt from the other of course restricted by the number of tokens in the respective pile.) Does the answer depend on the specific values of p and q? We let (p, q)GDWN, $0 < p < q$, denote this Generalized Diagonal extension of Wythoff Nim and $\mathcal{P}(p, q)$ its set of P-positions. (See Figure 1 for the rules of $(1, 2)$GDWN and its initial P-positions, $\mathcal{P}(1, 2)$.) Given such a game, we define the sequences $a = a(p, q) = (a_n)_{n \in \mathbb{N}}$ and $b = b(p, q) = (b_n)_{n \in \mathbb{N}}$ via

$$(2) \qquad\qquad \mathcal{P}(p, q) = \{(a_n, b_n), (b_n, a_n) \mid n \in \mathbb{N}\} \cup \{(0, 0)\}$$

where the ordered pairs of the form (a_n, b_n), the *upper* P-positions, are distinct, where $a_n \le b_n$, for all $n \ge 1$ and the sequence a is non-decreasing. For a technical reason we omit the terminal P-position $(a_0, b_0) = (0, 0)$ in the definition of a and b. Here we have used that, since the moves of (p, q)GDWN are 'symmetric' (that is (m_1, m_2) is a move if and only if (m_2, m_1) is), the P-positions are also. The purpose of this paper is to investigate some of the properties of the sequences a and b given in (2).

Our main theorem considers a necessary condition for 'splitting' upper P-beams for the case $(1, 2)$GDWN.

Theorem 1.1. *Let $\mathcal{P}(1, 2)$ define the sequences $a = a(1, 2)$ and $b = b(1, 2)$ as in (2). Then the limit*

$$\lim_{n \in \mathbb{N}} \frac{b_n}{a_n}$$

does not exist.

The reader who wants to study the proof of this theorem may at this point skip to Section 2. In order to state a conjecture concerning general (p, q)GDWN, let us introduce the following notation.
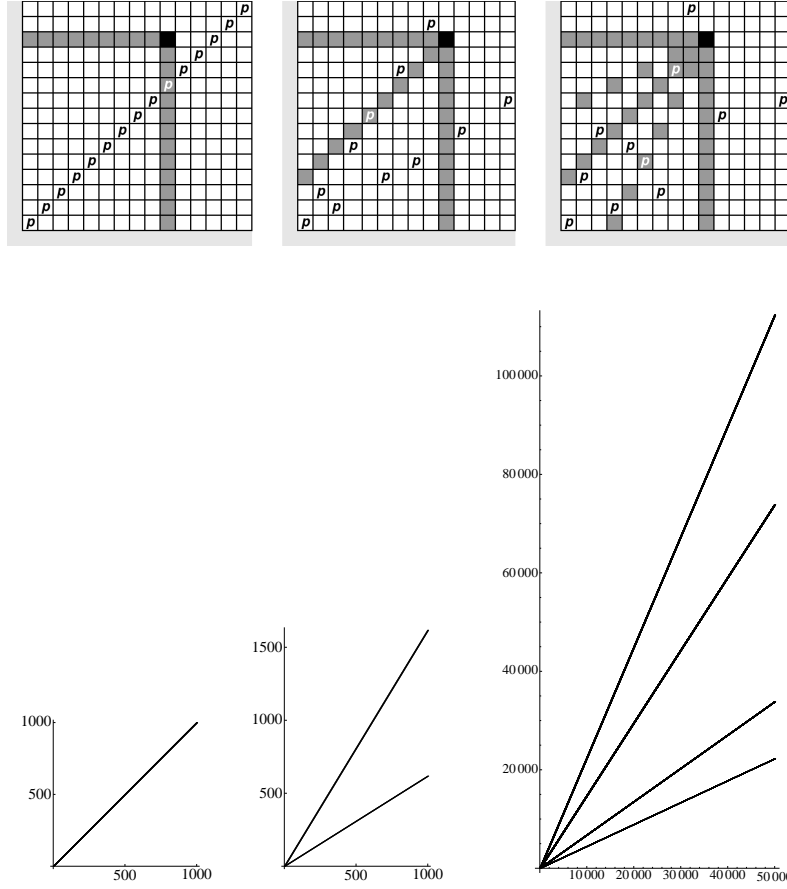
FIGURE 1. The upper figures illustrate typical moves (in dark gray) and initial P-positions of Nim, Wythoff Nim, and $(1, 2)$GDWN respectively. The black square is a given game position. The white 'P's represent the winning options from this position. The lower figures give a macroscopic view of the P-positions of the respective games. The left most figure illustrates 2-pile Nim's single P-beam of slope 1. Then, in the middle we illustrate Wythoff Nim's pair of P-beams with slopes ϕ and $1/\phi$ respectively and, at last, we present the P-positions of $(1, 2)$GDWN, where our computations, for all x-coordinates ≤ 50000, seem to suggest that each one of Wythoff Nim's P-beams has 'split' into two new distinct P-beams, See also Figures 2, 3 and 8 on pages 17, 18 and 22 respectively.

An ordered pair of positive integers belongs to the set $\mathcal{W}$ if it is of the form $(\lfloor \phi t \rfloor, \lfloor \phi^2 t \rfloor)$, a *Wythoff pair*, or of the form $(\lceil \phi t \rceil, \lceil \phi^2 t \rceil)$, a *dual Wythoff*

pair, $t \in \mathbb{N}$. Hence

$$\mathcal{W} = \{(1,2), (2,3), (3,5), (4,6), (4,7), (5,8), (6,10), (7,11), \dots\}.$$

Notice that, for all $t \in \mathbb{N}$, $\lceil \phi^2 t \rceil / \lceil \phi t \rceil < \phi$ and $\lfloor \phi^2 t \rfloor / \lfloor \phi t \rfloor > \phi$. In view of Section 3 and various experimental results in Section 4, we conjecture a continuation of Theorem 1 if and only if $(p, q) \in \mathcal{W}$.

Conjecture 1. *Let the sequences* a *and* b *be defined by* $(p, q)GDWN$, *as in (2), and with* $(p, q) \notin \mathcal{W}$. *Then, the limit*

$$(3) \qquad\qquad\qquad \lim_{n \in \mathbb{N}} \frac{b_n}{a_n}$$

exists and equals $\phi = \frac{1+\sqrt{5}}{2}$. *Otherwise, that is if* $(p, q) \in \mathcal{W}$, *the limit (3) does not exist.*

For a strengthening of this conjecture we give a rigorous definition of the term 'split'. To this purpose we need to extend our terminology accordingly. Let $\mu \in \mathbb{R}$ with $\mu > 0$. A sequence of pairs of positive integers $((x_i, y_i))_{i \in \mathbb{N}}$, with, for all i, $x_i \le y_i$, μ-*splits* if there is an $\alpha \in \mathbb{R}$ such that,

- there are at most finitely many i's such that $y_i/x_i \in [\alpha, \alpha + \mu)$,
- there are infinitely many i's such that $y_i/x_i \in [\alpha + \mu, \infty)$,
- there are infinitely many i's such that $y_i/x_i \in (0, \alpha)$.

We say that $((x_i, y_i))_{i \in \mathbb{N}}$ *splits* if there is a μ such that $((x_i, y_i))_{i \in \mathbb{N}}$ μ-splits. If $((x_i, y_i))_{i \in \mathbb{N}}$ splits we may take $\xi \in [\alpha, \alpha + \mu)$ and define complementary sequences (l_i) and (u_i) such that for all i

$$y_{l_i}/x_{l_i} \in (0, \alpha + \xi)$$

and

$$y_{u_i}/x_{u_i} \in [\alpha + \xi, \infty).$$

Clearly, the P-positions of Nim do not split, but the ones of Wythoff Nim do. Indeed, the latter 1-splits (with $\alpha = \phi - 1$). (See also [Lar11] for a discussion of splitting sequences in the context of Blocking Wythoff Nim.)

Since a sequence of P-positions of a $(p, q)GDWN$ game can 'split once', we wonder whether there is some $(p, q)GDWN$ such that the P-positions of the new game 'splits more than once'. (See also Section 5.) Suppose that $((x_i, y_i))_{i \in \mathbb{N}}$ splits and that α and μ are chosen so that μ is the largest possible. Suppose also that there is an interval $[\beta, \beta + \mu']$ such that $[\beta, \beta + \mu'] \cap [\alpha, \alpha + \mu) = \emptyset$ and such that

- there are at most finitely many i's such that $y_i/x_i \in [\beta, \beta + \mu')$, and either
- $\beta > \alpha + \mu$ and
    - there are infinitely many i's such that $y_i/x_i \in [\beta + \mu', \infty)$,
    - there are infinitely many i's such that $y_i/x_i \in [\alpha + \mu, \beta)$,
- or $\beta + \mu' < \alpha$, and
    - there are infinitely many i's such that $y_i/x_i \in [\beta + \mu', \alpha)$,
    - there are infinitely many i's such that $y_i/x_i \in [1, \beta)$,

then we say that $((x_i, y_i))_{i \in \mathbb{N}}$ *splits twice*. If $((x_i, y_i))_{i \in \mathbb{N}}$ splits, but does not split twice, we say that $((x_i, y_i))_{i \in \mathbb{N}}$ splits (precisely) *once*.

Conjecture 2. *The sequence of upper P-positions of $(p, q)GDWN$ splits precisely once if and only if $(p, q) \in \mathcal{W}$. Furthermore, if $(p, q) = (1, 2)$ or $(p, q) = (2, 3)$, then there is a pair of increasing complementary sequences (l_i) and (u_i) such that both $\eta = \lim_{i \to \infty} \frac{b_{l_i}}{a_{l_i}}$ and $\gamma = \lim_{i \to \infty} \frac{b_{u_i}}{a_{u_i}}$ exist with real $1 < \eta < \phi < \gamma \leq 3$.*

1.1. **Exposition.** The proof of Theorem 1.1 is studied in Section 2. Then, in Section 3, we prove a stronger form of the first part of Conjecture 1 for a whole subfamily of games, namely whenever $q/p < \phi$ and (p, q) is not a dual Wythoff pair. In Section 4 we support our conjectures by experimental data and figures. Finally, in Section 5 we discuss some further directions for future research.

2. A RESOLUTION OF THEOREM 1.1

Two sequences of positive integers are said to be *complementary*, e.g. [Fra82], if each positive integer occurs precisely once in precisely one of these sequences.

Proposition 2.1. *As in (2), let $\mathcal{P}(p, q) = \{(a_i, b_i), (b_i, a_i)\} \cup \{(0, 0)\}$ define the sequences a and b, of which a is non-decreasing and, for all $n \in \mathbb{N}$, $a_n \leq b_n$. Then*

   (i) $a_1 = 1$,
   (ii) $a_i < b_i$, *for all $i > 0$*,
   (iii) $a_i \neq a_{i+1}$, *for all i, that is a is strictly increasing*,
   (iv) $a_i \neq b_j$ *for all $i > 0$ and $j > 0$*,
   (v) *for each $n \in \mathbb{N}$ there is an i such that either $a_i = n$ or $b_i = n$*,

that is, a and b are complementary.

Proof. By $(0, 0) \in \mathcal{P}(p, q)$ and by the Nim-type moves, there can be no P-position of the form $(0, x), x \in \mathbb{N}$. For (i), notice that there is a least x such that $(1, x)$ is P. Namely, if $(p, q) = (1, 2)$ then $x = 3$, otherwise $x = 2$. Clearly (ii) follows from (iv), but it makes sense to start with (ii). Thus, suppose $a_i = b_i$ for some $i > 0$. Then $(a_i, b_i) \to (0, 0)$ is a legal diagonal-type move, which contradicts the definition of P. For (iii), suppose that $a_i = a_{i+1}$ for some i. Then, by (i), $i > 0$ and so either $(a_{i+1}, b_{i+1}) \to (a_i, b_i)$ or $(a_i, b_i) \to (a_{i+1}, b_{i+1})$ is a legal Nim-type move, but either case is ridiculous by definition of P. For (iv), suppose that there were integers $i > j > 0$ such that $a_i = b_j$. Then $(a_i, b_i) \to (b_j, a_j)$ is a Nim-type move since, by (ii), $b_i > a_i = b_j > a_j$, a contradiction. For (v), let $n \in \mathbb{N}$. Then, each position of the form

$$(4) \qquad (a_m, b_m), a_m < n \text{ or } (b_m, a_m), b_m < n$$

can, by the rules of (p,q)GDWN, be reached by at most four positions of the form $(n, \cdot)$. By (iii) and (iv), there are at most n positions of the form in (4). We conclude that there exists a least $x \leq 4n$ such that the position (n, x) does not have a P-position as an option. Then (n, x) is P and hence of the desired form.   □

Let $\mathbb{R}$ denote the real numbers and let $f, g : \mathbb{N}_0 \to \mathbb{R}$. We use the notation $f(N) \ll g(N)$ if $f(N) < g(N)$ for all sufficiently large N. (And analogously for $\gg$), where the term sufficiently large is explained by each surrounding context.

We have use for a well-known and general result (often phrased in terms of bipartite graphs). Simple as it is, it turns out to be a very useful tool.

Lemma 2.2. *Let $A = \{A_i\}$ and $B = \{B_i\}$ denote sets of non-negative integers satisfying $A_i < A_{i+1}$, $A_i \leq B_i$, for all $i \in \mathbb{N}$ and, for each $N \in \mathbb{N}$, there is precisely one i such that*

(5) $$A_i = N \ \text{or} \ B_i = N,$$

(possibly both). Then, for all $N \in \mathbb{N}$,

$$\#(A \cap \{1, \ldots, N\}) \geq \frac{N}{2}.$$

Proof. Put $n = n(N) := \#(A \cap \{1, \ldots, N\})$ and note that $A_n \leq N < A_{n+1}$. Suppose on the contrary that

(6) $$2n < N$$

for some $N \in \mathbb{N}$. By definition, $B_{n+1} \geq A_{n+1} > N$, so that there are at most n numbers from the B-sequence less than N. This gives

$$2n \geq \#(A \cap \{1, \ldots, N\}) + \#(B \cap \{1, \ldots, N\}) \geq N > 2n,$$

where the last inequality is by (6).   □

Note that our sequences a and b satisfy the conditions of (A_i) and (B_i) in Lemma 2.2. The next lemma deepens this result to another powerful tool for our purposes. Roughly spoken, it concerns the structure of P-positions for 'extensions' of Wythoff Nim.

Lemma 2.3. *Let $A = (A_i)$ and $B = (B_i)$ denote complementary sequences of positive integers satisfying, for all $i, j \in \mathbb{N}$, $A_i < A_{i+1}$, $A_i \leq B_i$ and*

(7) $$B_i - A_i = B_j - A_j \ \text{if and only if} \ i = j.$$

Then the set

$$\{i \mid B_i > \phi A_i\}$$

is infinite.

Proof. Let $C \in \mathbb{R}$ with $C > 1$ and define the set

$$S = S(C) := \{i \mid B_i < CA_i\}.$$

For all i, define
$$\delta_i := B_i - A_i \geq 0.$$
Suppose that S contains all but finitely many numbers and let

(8) $$n > \max\{\delta_i \mid i \in (\mathbb{N}_0 \setminus S)\}.$$

By definition of A and B, there is an $x \leq n$ such that $\delta_x \geq n$. Together with the definition of S this gives,
$$1 + \frac{n}{A_x} = \frac{A_x + n}{A_x} \leq \frac{B_x}{A_x} < C,$$
so that

(9) $$n \ll (C-1)A_x \leq (C-1)A_n,$$

since (A_i) is increasing. On the other hand, by Lemma 2.2, we have that $n \geq \frac{A_n}{2}$, so that we may conclude that $C > \frac{3}{2}$. Denote with $c = C - 1 > \frac{1}{2}$.

For sufficiently large n, by (9) and complementarity, the number of i's such that $B_i < A_n$ is

(10) $$A_n - n \gg (2 - C)A_n = (1 - c)A_n,$$

which, by (7) gives that, there is a least $j \leq n$, such that

(11) $$\delta_j \geq (1 - c)A_n.$$

We may ask, where is this least j? Define the set
$$\rho = \rho(c, n) := \{i \mid A_i \leq cA_n\}.$$

Case 1: If $j \in \rho$, then
$$C \gg \frac{B_j}{A_j} = 1 + \frac{\delta_j}{A_j} \geq 1 + \frac{1 - c}{c}$$
which is equivalent to
$$C^2 > C + 1,$$
which holds if and only if $C > \phi$.

Case 2: If $j \notin \rho$, then since B_j is the least number in the B-sequence such that (11) holds, for $i < j$, we get $B_i < (1 - c)A_n + A_i \leq A_n$. But then, by (10), $(1 - c)A_n \leq \max \rho$. By applying the same argument as in (9), we also have that
$$\max \rho \ll c^2 A_n$$
and so, again, $C > \phi$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For the rest of this section, we let $(a_i) = a(1, 2)$ and $(b_i) = b(1, 2)$.

Proposition 2.4. *Let $R := \{b_i/a_i \mid i \in \mathbb{N}\}$. Then the following three items hold.*

    (i) The set $(\phi, \infty] \cap R$ is infinite.

    (ii) Let $C < 2 < D$ denote two constants with $\beta := D - C < 1/2$. Then $([1, C) \cup (D, \infty)) \cap R$ is infinite.

(iii) The set $[1, 2] \cap R$ is infinite.

Proof. Item (i) follows since, by Proposition 2.1, the sequences a and b satisfy the conditions of A and B in Lemma 2.3, respectively. (The condition (7) is satisfied, since otherwise a removal of the same number of tokens from each heap would connect two P-positions, which is impossible.)

For item (ii), suppose, for a contradiction, that all but finitely many ratios from R lie in $[C, D]$ and define

$$r := \#\{i \mid b_i/a_i \notin [C, D]\}.$$

Suppose now that $b_i/a_i \in [C, D]$ with $a_i \leq N$, $N \in \mathbb{N}$. Then we get

(12) $$2(N - a_i) + b_i \in I(N) := [CN, DN].$$

The upper bound follows from $b_i - 2a_i \leq Da_i - 2a_i \leq (D - 2)N$ and the lower is similar.

Denote by $J(N)$ the number of pairs (a_i, b_i) with $a_i \leq N$ such that $b_i/a_i \in [C, D]$. Then, by Lemma 2.2 and definition of r, for all $N > r$,

$$J(N) \geq \frac{N}{2} - r.$$

This gives that, for all $\epsilon > 0$, for all sufficiently large $N = N_\epsilon$, we have that

(13) $$\frac{J(N) - 1}{N} \geq \frac{1}{2} - \frac{r + 1}{N} > \frac{1}{2} - \epsilon.$$

In particular we may take $\epsilon := 1/2 - D + C > 0$ and define N as N', a fixed integer strictly greater than $\frac{2(r+1)}{1 - 2(D-C)}$. The number of integer points in $I(N')$ is

$$\lceil DN' \rceil - \lceil CN' \rceil.$$

If we divide this expression with N' and compare with our definition of ϵ, we get

$$\frac{\lceil DN' \rceil - \lceil CN' \rceil}{N'} \leq \frac{1}{2} - \epsilon + \frac{1}{N'}.$$

Hence, by (13), we get that the number of integer points in the interval $I(N')$ is

(14) $$\lceil DN' \rceil - \lceil CN' \rceil < J(N').$$

Observe that each pair (a_i, b_i), which is counted by $J(N')$, defines a line of slope 2 which intersects the line $x = N'$ at an integer point, which resides inside the interval $I(N')$. Thus, by (14), the Pigeonhole principle gives that, for some integer $t \in I(N')$, there exists a pair $i < j < N'$ such that

$$t = 2(N' - a_i) + b_i$$
$$= 2(N' - a_j) + b_j.$$

But then $2(a_j - a_i) = b_j - b_i$ so that, by the definition of $(1, 2)$GDWN, there is a move $(a_j, b_j) \to (a_i, b_i)$, which contradicts definition of P. This proves item (ii).

Let us proceed with item (iii). We begin by proving two claims.

Claim 1. Let $N \in \mathbb{N}$ be such that $b_N \geq 2a_N$. Clearly, by Table 1, there is such an N. Then, if there exists a least $k \in \mathbb{N}$ such that $b_{N+k} > 2a_{N+k}$, it follows that $b_{N+k} - 2a_{N+k} = b_N - 2a_N + 1$. (By Table 1 the first such case is $N = 0, k = 1$ and the "γ-row" gives an initial sequence of pairs "(N_i, k_i)" as follows: $(0,1)(1,1)(2,5)(7,1)(8,2)(10,4)(14,2)(16, k_8)$.)

b_n	0	3	6	5	10	14	17	25	28	18	35	23	31	29	48	32
a_n	0	1	2	4	7	8	9	11	12	13	15	16	19	20	21	22
δ_n	0	2	4	1	3	6	8	14	16	5	20	7	12	9	27	10
γ_n	0	1	2	-3	-4	-2	-1	3	4	-8	5	-9	-7	-11	6	-12
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

TABLE 1. Here (a_n, b_n) represents a P-position of $(1,2)$GDWN for $0 \leq n \leq 18$. Further, $\delta_n = b_n - a_n$ and $\gamma_n = b_n - 2a_n$.

Proof of Claim 1. Suppose that $N > 1$ is chosen smallest possible such that, unlike the assumption, there is a least $k > 0$ such that

$$\frac{b_{N+k}}{a_{N+k}} > 2,$$

and $b_{N+k} - b_N \neq 2(a_{N+k} - a_N) + 1$. Then, by the minimality of N, we must have

$$(15) \qquad b_{N+k} - 1 - 2a_{N+k} > b_N - 2a_N,$$

which, by Proposition 2.1, implies that $(a_{N+k}, b_{N+k} - 1)$ is N. That is, there exists a $j < N$ such that

$$b_{N+k} - 1 - b_j = \gamma(a_{N+k} - a_j),$$

where $\gamma = 0, \frac{1}{2}, 1$ or 2. Put

$$y := b_N - 2(a_N - a_j).$$

Altogether, by (15), we get

$$b_j - y = (2 - \gamma)(a_{N+k} - a_j) + 1 > 0.$$

But, by minimality of N, b_j must be strictly less than y, a contradiction.

From this point and onwards we assume that $[1,2] \cap R$ is finite (and hence we are going to find a contradiction). The next claim concerns a consequence of this assumption combined with the result in Claim 1.

Claim 2. Suppose that $[1, 2] \cap R$ is finite. Then there is an $r \in \mathbb{N}$ such that, for all $N \geq r$, we have

$$(16) \qquad b_{N+1} - b_N = 3 \text{ and } a_{N+1} - a_N = 1$$

or

$$(17) \qquad b_{N+1} - b_N = 5 \text{ and } a_{N+1} - a_N = 2.$$

Proof of Claim 2. Since $[1, 2] \cap R$ is finite, we get that, for some $s \in \mathbb{N}$, for all $j \geq s$, $b_j/a_j > 2$. By Claim 1, since (a_i) is increasing, this implies that $b_{j+1} \geq b_j + 3$. Further, by definition of a_{N+1}, if N is such that $a_N \geq b_s$, this gives

$$(18) \qquad a_{N+1} - a_N \leq 2.$$

Plugging this into the result of Claim 1 we get either (16) or (17). This first part of the proof of Claim 2 implies that both a and $(b_i)_{r \leq i}$ are strictly increasing. Then, by complementarity of a and b it follows that ($\star$) there are infinitely many N's such that (17) holds.

The remainder of the proof consists of a geometric argument contradicting the 'greedy' definition of the b-sequence. We show (implicitly) that there would be an N-position too much if (iii) fails to hold.

By Claim 2, we can find $r < u < v$ such that (17) holds for both $b_u < b_v$. Define four lines accordingly:

$$l_u(x) = x + b_u,$$
$$l_{u+1}(x) = x + b_u + 3,$$
$$l_v(x) = b_v,$$
$$l_{v+1}(x) = b_v + 5,$$

These fours lines intersect at the positions (lattice points) $((\alpha_i, \beta_i))_{i \in \{1,2,3,4\}} = ((b_v - b_u - 3, b_v), (b_v - b_u, b_v), (b_v - b_u + 2, b_v + 5), (b_v - b_u + 5, b_v + 5))$, defining the corners of a parallelogram. Denote the set of positions *strictly* inside this parallelogram by $\mathcal{K}$. Then, by inspection

$$\#\mathcal{K} = 8$$

and, by ($\star$), we may assume that we have chosen v sufficiently large so that, for all $(x, y) \in \mathcal{K}$,

$$(19) \qquad 1 < \frac{y}{x} < 2.$$

Denote by $\mathcal{L}$ another set of lines satisfying the following conditions. A line l belongs to $\mathcal{L}$ if and only if:

(a) Its slope is either $1/2, 2$ or ∞.
(b) It intersect a point of the form (a_s, b_s) or (b_s, a_s) with $s \geq r$.
(c) It intersects $\mathcal{K}$.

By the definition of $\mathcal{K}$ it follows from (17) that i and j may be defined such that each line of form (b) and (c) is also of the form (a). Again, by ($\star$) we may assume that we have chosen i and j sufficiently large so that the first part of (b) together with (c) implies $s \geq r$.

Claim 3: There is a game position (lattice point) in the set $\mathcal{K} \setminus \mathcal{L}$.

Clearly, by the definition of (b_i) and by (19), this claim contradicts the assumption that $[1,2] \cap R$ is finite. (In fact it would imply the existence of an N-position in $\mathcal{K}$ without a P-position as a follower.)

Proof of Claim 3. Let

$$\mathcal{K}' := \{(0,0), (1,0), (1,1), (2,1), (2,2), (3,2), (3,3), (4,3)\}.$$

Then $\mathcal{K}'$ is simply a linear translation of $\mathcal{K}$. (Namely, given $(x,y) \in \mathcal{K}$, $T(x,y) = x - (b_j - b_i - 1), y - (b_j + 1) \in \mathcal{K}'$.)

Let $\alpha \in \mathbb{R}$. Clearly, the two lines $x + \alpha$ and $x + 3 + \alpha$ can together cover at most three points in $\mathcal{K}'$, namely choose $\alpha = 0$ or 1. The two lines $2x - \alpha$ and $2x - 5 - \alpha$ can cover at most two points in $\mathcal{K}'$, namely we may choose $\alpha = 0, 2$ or 3. (In fact, for the two latter cases it is only the former line that contributes.) On the other hand, the two lines $x/2 + \alpha$ and $x/2 + 5/2 + \alpha$ can cover at most two points in $\mathcal{K}'$, namely, if we choose $\alpha = 0, 1/2$ or 1. (In fact, for these α, it is only the former line that contributes.)

Fix any set of the above six lines, depending only on the choices of α for the respective cases, and denote this set by $\mathcal{L}'$. Then, as we have seen, $\#(\mathcal{L}' \cap \mathcal{K}') \leq 7$. But, by Claim 2, an instance of $\mathcal{L} \cap \mathcal{K}$ is simply a linear translation of some set $\mathcal{L}' \cap \mathcal{K}'$. The claim follows and so does the proposition. $\square$

We are now well prepared to conclude what has been accomplished so far.

Proof of Theorem 1.1 Suppose on the contrary that $\alpha := \lim_{n \in \mathbb{N}} \frac{b_n}{a_n}$ exists. Then either

(a) $\alpha \in [1, \phi)$,
(b) $\alpha \in [\phi, 2]$, or
(c) $\alpha \in (2, \infty]$.

By Proposition 2.4 (i), (a) is impossible. On the other hand (b) is contradicted by Proposition 2.4 (ii) with, say, $C = \phi$ and $D = 2$. For the last case Proposition 2.4 (iii) gives a contradiction. $\square$

Remark 1. *It is clear that Proposition 2.4 (i) may be adapted without any changes for general (p, q)'s. As it stands, however, the combined ideas in (i), (ii) and (iii) do not seem to suffice to prove an actual split of the upper P-positions even for the $(1, 2)$ case. Therefore we have chosen not to pursue*

a generalization of (ii) and (iii) at this point. An interesting observation in support of Conjecture 2 is that $(1,2)GDWN$ splits if a positive proportion of the ratios b_i/a_i is greater than 2, that is if the set $\{i \in \mathbb{N} \mid b_i/a_i > 2\}$ has positive density. This follows from an argument similar to that of the proof of Proposition 2.4 (iii).

In [LaWä] a restriction, called Maharaja Nim, of the game $(1,2)$GDWN is studied, where precisely the "Knight-type" moves $(1,2)$ and $(2,1)$ are adjoined to the game of Wythoff Nim. In contrast to the main result of this paper, for Maharaja Nim it is proved that the P-positions lie on the same 'beams' as in Wythoff Nim. In fact, in [LaWä], we prove that the quotients of the coordinates of the P-positions of Maharaja Nim lie withing a bounded distance from the lines $\phi^{-1}x$ and ϕx respectively.

In this context it is interesting to observe that the only move on the generalized diagonal $(t, 2t)$ which belongs to $\mathcal{W}$ is $(1,2)$. Viewed in a slightly different perspective, we have proved that, for $(1,2)$GDWN, the non-$\mathcal{W}$ pairs on the diagonals $(t, 2t)$ and $(2t, t)$, $t > 1$, contribute significantly in destroying the asymptotes of the P-positions of Maharaja Nim. For yet another perspective of these type of questions, see Table 2 and Figure 4 for the game $(2,4)$GDWN.

3. More on the $\mathcal{W}$-pairs

The main result of this section is the following partial resolution of Conjecture 1.

Proposition 3.1. *Suppose that $(p, q) \notin \mathcal{W}$. Then $\mathcal{P}(p, q) = \mathcal{P}(WN)$ if $1 < \frac{q}{p} < \phi$. That is, with $a = a(p, q)$ and $b = b(p, q)$, for all $n \in \mathbb{N}$, $a_n = \lfloor \phi n \rfloor$ and $b_n = \lfloor \phi^2 n \rfloor$.*

Before proving this proposition we need to develop some facts from combinatorics on Sturmian words. For some background and terminology on this subject we refer to [Lot01, Section 1 & 2].

We are interested in the (infinite) Sturmian words s and s' on the alphabet $\{0, 1\}$ and the corresponding (non-Sturmian) 'translates' t and t' on $\{1, 2\}$. For all $n \in \mathbb{N}_0$, the nth letter is

$$s(n) := \lfloor \phi(n+1) \rfloor - \lfloor \phi n \rfloor - \lfloor \phi \rfloor,$$
$$s'(n) := \lceil \phi(n+1) \rceil - \lceil \phi n \rceil - \lfloor \phi \rfloor,$$
$$t(n) := \lfloor \phi(n+1) \rfloor - \lfloor \phi n \rfloor = s(n) + 1$$

and

$$t'(n) := \lceil \phi(n+1) \rceil - \lceil \phi n \rceil = s'(n) + 1$$

respectively.

Then s and t (s' and t') are the lower (upper) mechanical words with *slopes* $1/\phi$ and ϕ respectively and intercept 0. An *irrational mechanical*

word has irrational slope. The *characteristic* word belonging to s and s' is $c = s(1)s(2)s(3)\ldots$. Namely, we have $s(0) = 0$, $s'(0) = 1$ and otherwise, for all $n > 0$, $s(n) = s'(n) = 0$ or $s(n) = s'(n) = 1$. In fact, we have

$$s = 01011010110\ldots$$

and

$$s' = 11011010110\ldots.$$

Let x denote a finite word on $\{0, 1\}$. Then $l(x)$ and $h(x)$ denote the number of letters and 1's in x, respectively. Let α and β be two *factors* of a Sturmian word w. Then w is *balanced* if $l(\alpha) = l(\beta)$ implies $\mid h(\alpha) - h(\beta) \mid \leq 1$. By [Lot01, Section 2], both s and s' are balanced (aperiodic) words. We will also need the following result from the same source.

Lemma 3.2 ([Lot01]). *Suppose two irrational mechanical words have the same slope. Then their respective set of factors are identical.*

We also use the following notation. Let $x = x_1 x_2 \ldots x_n$ be a factor of a mechanical word on n letters. Then we define the *sum* of x as $\sum x := x_1 + x_2 + \ldots + x_n$. For example the sum of 2121 equals 6. We let $\xi_n(r)$ denote the unique n-letter *prefix* of an infinite word r. Note that $\sum \xi_n(t) = \sum \xi_n(s) + n$ and $\sum \xi_n(t') = \sum \xi_n(s') + n$, for all n.

Lemma 3.3. *Let x be any factor of s (or s'). Then*

$$\sum x = \sum \xi_{l(x)}(s) \ or \ \sum x = \sum \xi_{l(x)}(s').$$

Proof. If two factors of s have the same length and the same height, then, since the number of 1's in the respective factors must be the same, their sums are identical. Therefore, if $h(x) = h(\xi_{l(x)}(s))$, this implies $\sum x = \sum \xi_{l(x)}(s)$.

Assume on the contrary that $h(x) \neq h(\xi_{l(x)}(s))$. On the one hand, for all n, $h(\xi_n(s)) = h(\xi_n(s')) - 1$. On the other hand, the balanced condition implies that if x is a factor of s with a given length, then $h(x)$ takes the value of one of two consecutive integers. It follows that $h(x) = h(\xi_{l(x)}(s'))$. But then, by the initial observation, we are done. □

The following proposition assures that for each pair in $\mathcal{W}$, a 'split is initiated'. (On the other hand, Table 2 and Figure 4 give examples where a split of P-positions seems to be initiated, but where experimental results suggest that the ratio of the coordinates of the upper P-positions still converge to ϕ.) An alternative proof appears in [DFNR].

Proposition 3.4. [DFNR] *Let $p, q \in N$. Then $(p, q) \in \mathcal{W}$ if and only if there exists a pair $m, n \in \mathbb{N}_0$ with $m < n$ such that*

$$(p, q) = (\lfloor \phi n \rfloor - \lfloor \phi m \rfloor, \lfloor \phi^2 n \rfloor - \lfloor \phi^2 m \rfloor).$$

Proof. Let $(p, q) \in \mathcal{W}$. If $(p, q) = (\lfloor r\phi \rfloor, \lfloor r\phi^2 \rfloor)$, for some $r \in \mathbb{N}$, we may take $m = 0$. If $(p, q) = (\lfloor r\phi \rfloor + 1, \lfloor r\phi^2 \rfloor + 1)$, for some $r \in \mathbb{N}$, then, since s and s' are mechanical with the same slope, by Lemma 3.2, $\xi_r(s')$ is a factor of s.

But then, $\lfloor r\phi \rfloor + 1 = \sum \xi_r(t') = \sum \xi_n(t) - \sum \xi_m(t)$ for some $n - m = r$. This gives $(p, q) = (\lfloor r\phi \rfloor + 1, \lfloor r\phi \rfloor + 1 + r) = (\lfloor n\phi \rfloor - \lfloor m\phi \rfloor, \lfloor n\phi \rfloor - \lfloor m \rfloor + n - m) = (\lfloor n\phi \rfloor - \lfloor m\phi \rfloor, \lfloor \phi^2 n \rfloor - \lfloor \phi^2 m \rfloor)$. For the other direction, let (p, q) and $m < n$ be as in the proposition. Let x denote the factor of s which consists of the $n - m$ last letters in the prefix $\xi_n(s)$. Then, by $l(x) = n - m$ and Lemma 3.3, we may take $p = \sum \xi_{l(x)}(t) = \lfloor l(x)\phi \rfloor$ or $p = \sum \xi_{l(x)}(t') = \lfloor l(x)\phi \rfloor + 1$. In either case, the assumption gives $q = p + l(x)$, and so $(p, q) \in \mathcal{W}$. $\square$

Proof of Proposition 3.1. We need to show that, for all n, $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) = (a_n, b_n)$, where $a = a(p, q)$ and $b = b(p, q)$ and where $(p, q) \notin \mathcal{W}$ with $1 < p/q < \phi$. The problem of, for each candidate N-position, finding a move of (p, q)GDWN to a candidate P-position was resolved already in [Wyt07], namely it suffices to use the Wythoff Nim type moves. In order to assure that no two P-positions can be connected by a move it suffices to use Proposition 3.4. Namely, we proved, in particular, that there exist integers $0 \le m < n$ such that $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) \to (\lfloor \phi m \rfloor, \lfloor \phi^2 m \rfloor)$ is a legal move of (p, q)GDWN only if $(p, q) \in \mathcal{W}$.

Suppose that there are integers $0 \le m < n$ such that $(\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor) \to (\lfloor \phi^2 m \rfloor, \lfloor \phi m \rfloor)$ is a legal move of (p, q)GDWN. Then $(\lfloor \phi n \rfloor - \lfloor \phi^2 m \rfloor, \lfloor \phi^2 n \rfloor - \lfloor \phi m \rfloor) = (tp, tq)$ for some $t \in \mathbb{N}$. Hence

$$\lfloor \phi n \rfloor = tp + \lfloor \phi^2 m \rfloor$$

and

$$\lfloor \phi^2 n \rfloor = tq + \lfloor \phi m \rfloor.$$

Then $\phi < \lfloor \phi^2 n \rfloor / \lfloor \phi n \rfloor = (tq + \lfloor \phi m \rfloor)/(tp + \lfloor \phi^2 m \rfloor) < q/p$ since $0 \le \lfloor \phi m \rfloor \le \lfloor \phi^2 m \rfloor$ for all m. $\square$

4. Experiments, Conjectures and figures

In this section we support our conjectures with data for all games (p, q)GDWN with $p < q \le 9$. By Section 3, it suffices to investigate the cases for which $q/p > \phi$. Thus, we have explored several thousands of the upper P-positions for the 20 games (p, q)GDWN, where

$$(20) \quad (p, q) \in \{(1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 9), (2, 4), (2, 5),$$
$$(2, 6), (2, 7), (2, 8), (2, 9), (3, 6), (3, 7), (3, 8), (3, 9), (4, 8), (4, 9), (5, 9)\}.$$

This set consists exclusively of non-$\mathcal{W}$ pairs. Hence, to support Conjectures 1 and 2, we need to show experimental results pointing at a convergence of b_i/a_i to ϕ. Our question is: Given (p, q), is there a reasonably "small" $N_0 \in \mathbb{N}$ such that

$$(21) \qquad \frac{b_i}{a_i} \in [1.6175, 1.6185] \text{ for all } i \in [N_0, N_0 + 2000]?$$

(For each (p, q) if we find such an N_0 we stop the computation.) Our simulations provided a positive answer with $N_0 \le 10000$ for all (p, q) as in (20),

except $(1,3), (2,4), (3,6), (5,9)$. If we would have asked the same question, but with the upper bound in (21) exchanged for 1.623, then the answer would have been positive, with $N_0 = 15000$, also for these four pairs. In Figure 4 we display the results for the pair with slowest convergence in the test (20), namely $(2,4)$. (In this figure we also display the corresponding ratios for the game $(7,12)$GDWN which also appear to converge, but in an even slower rate.)

Remark 2. *For the main result in Section 4, since $q/p < \phi$, we did not need to consider the possibility of a connection between upper and 'lower' P-positions of Wythoff Nim. On the other hand, whenever $q/p > \phi$, by our simulations, it appears that such interferences are common. See Table 2 for the first such connection for the game $(2,4)$GDWN. It would be interesting to know whether infinitely many P-positions of $(2,4)$GDWN are Wythoff pairs. In case the answer is positive, is there an $n_0 \in \mathbb{N}$ such that, for all $y \ge x \ge n_0$, (x,y) is an upper P-position, if and only it is a Wythoff pair? In this context it would also be interesting to know whether each pair of integers (x,y) with $0 < x < y$ satisfies the equality $(tx, ty) = (\lfloor \phi m \rfloor - \lfloor \phi^2 n \rfloor, \lfloor \phi^2 m \rfloor - \lfloor \phi n \rfloor)$ infinitely often for some $m, n, t \in \mathbb{N}$ (and $m > n$). (These type of questions are of course relevant for any game (p,q)GDWN satisfying $q/p > \phi$.)*

b_n	0	2	5	7	10	17	14	19	18	20	27	33
a_n	0	1	3	4	6	8	9	11	12	13	16	21
δ_n	0	1	2	3	4	9	5	8	6	7	11	12
n	0	1	2	3	4	5	6	7	8	9	10	11

TABLE 2. Here (a_n, b_n) represent P-positions of $(2,4)$GDWN. Notice that $(8,13) - (2,1) = (3 \times 2, 3 \times 4)$, so that $(8,13)$ is the first Wythoff pair for which there is a move to a P-position of $(2,4)$GDWN. However, this type of connection of a Wythoff pair to a P-position does not seem to enforce a later split. Our computations rather suggest that the quotient b_n/a_n converges to ϕ in accordance with Conjecture 1 and Figure 4.

We also support our conjectures via computations of initial ratios b_i/a_i for some $\mathcal{W}$-pairs in Figures 2, 3 and 5, namely

$$(p,q) \in \{(1,2), (2,3), (3,5), (4,6), (4,7), (5,8), (6,10), (7,11)\}.$$

In Table 3 we conjecture μ-splits for the corresponding upper P-positions. See also Figures 2 and 3 for more specific conjectures on the games $(1,2)$GDWN and $(2,3)$GDWN.

Then, in Figures 6 and 7, we display some more exotic games (where, the game $(31,52)$GDWN passes the test in (21) with $N_0 = 5000$, but

(p, q)	μ	lower	upper	i'	total
$(1, 2)$	0.73	1.49	2.22	844	31523
$(2, 3)$	0.29	1.42	1.71	346	21768
$(3, 5)$	0.17	1.57	1.74	118	20565
$(4, 6)$	0.10	1.49	1.59	504	22807
$(4, 7)$	0.12	1.64	1.76	734	22170
$(5, 8)$	0.06	1.58	1.64	167	21237
$(6, 10)$	0.07	1.62	1.69	910	20962
$(7, 11)$	0.06	1.56	1.61	570	21256

TABLE 3. We conjecture μ-splits for (p, q)GDWN corresponding to column 1. The index i' indicates the greatest known index such that $b_{i'}/a_{i'}$ lies in between the 'lower' and 'upper' bounds for the conjectured μ-split. The ' total' column gives the number of computed upper P-positions.

$(31, 51)$GDWN seems so to converge much slower. (It does satisfy (21) for $\approx$ 99% of the ratios for $N_0 = 13000$.) On the other hand, in support of our conjectures, the games $(31, 50)$GDWN, $(32, 52)$GDWN and $(731, 1183)$GDWN violate the test in (21) a lot, as far as we have been able to compute.)
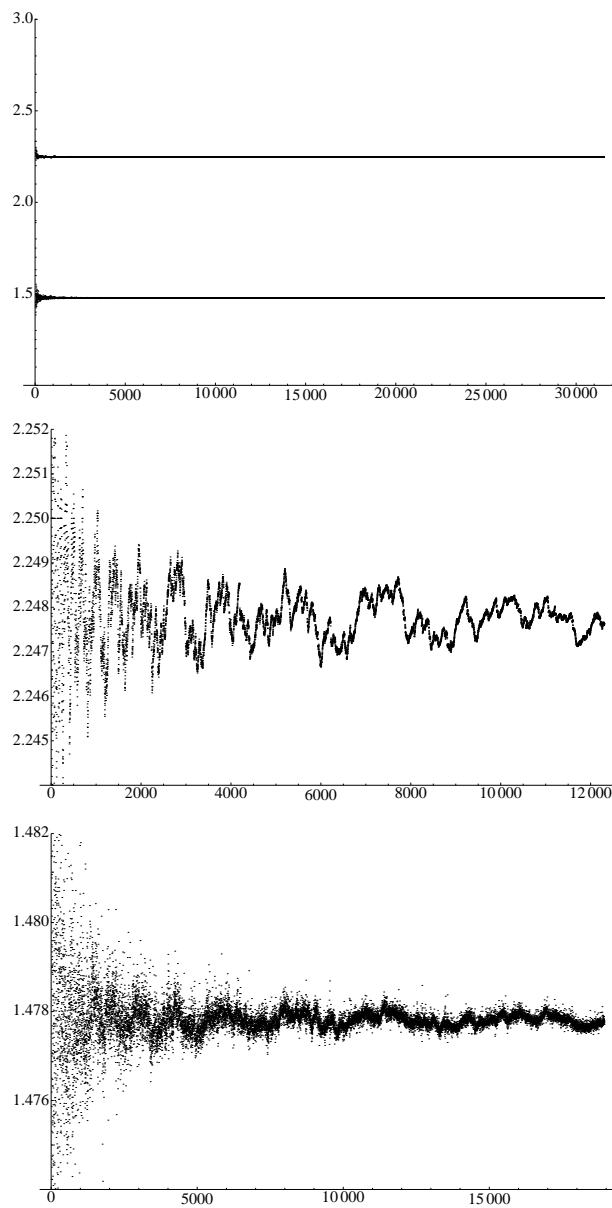
FIGURE 2. The upper figure illustrates the first ≈ 32000 ratios b_n/a_n for $(1,2)$GDWN. We conjecture that there exist two complementary sequences u (lower left) and l (lower right) such that $b_{u_i}/a_{u_i} \rightarrow 2.248\ldots$ (roughly 40%) and $\lim_{i\to\infty} b_{l_i}/a_{l_i} \rightarrow 1.478\ldots$ (roughly 60%). (The x-axes in the lower figures are relabeled corresponding to the new sequences.)
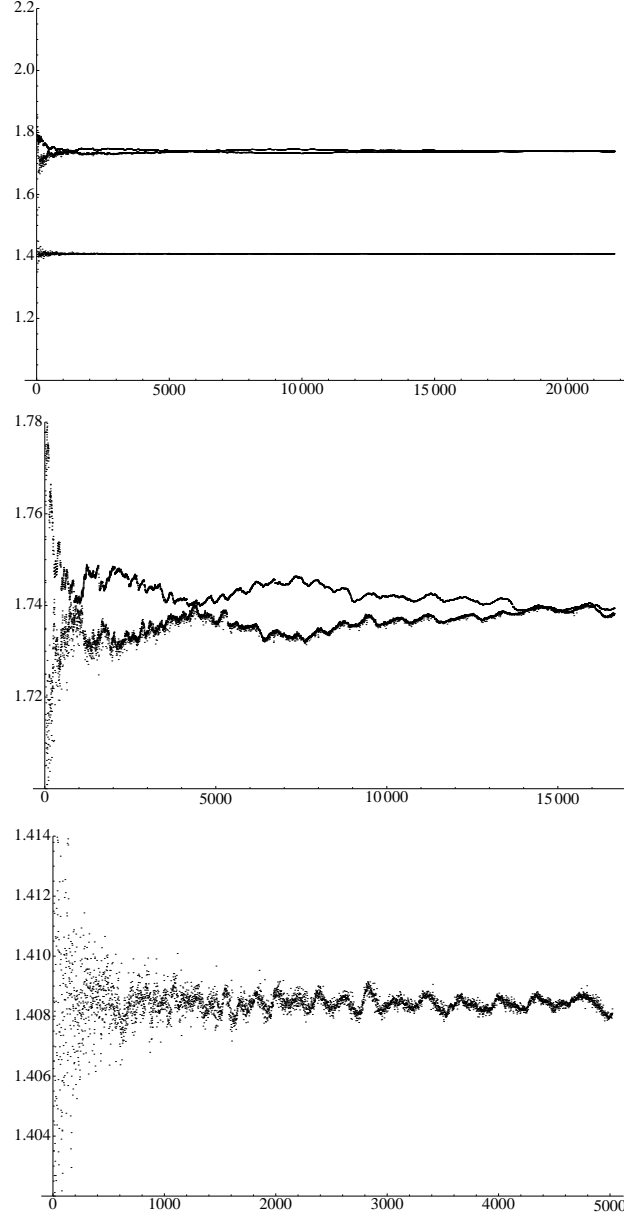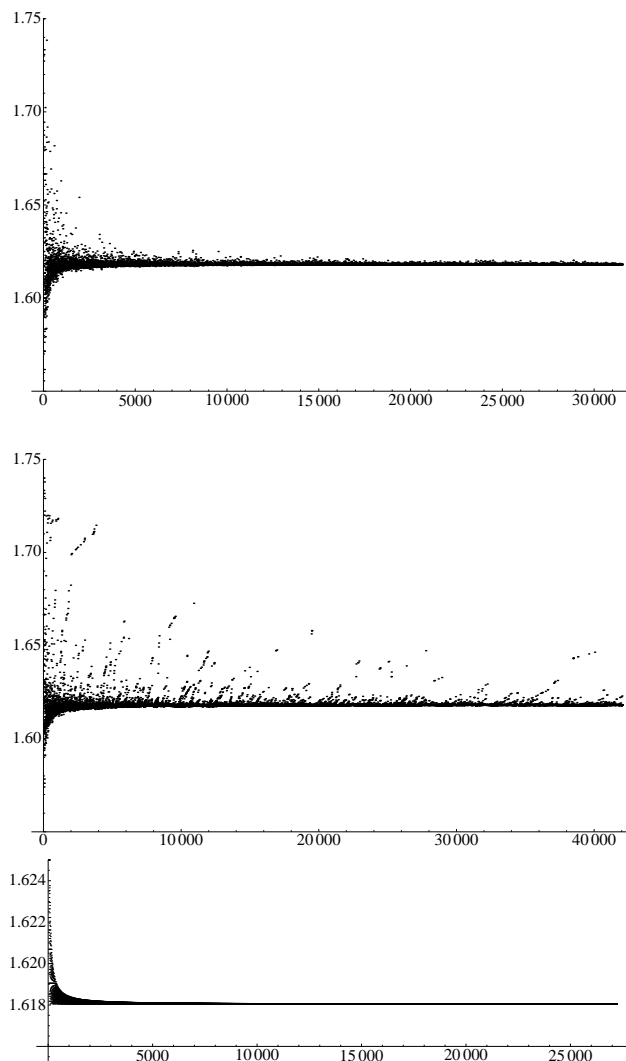
FIGURE 3. The first $approx 22000$ ratios b_n/a_n for $(2,3)$GDWN. We conjecture that there is a pair of complementary sequences u (lower left figure) and l (lower right figure) such that $b_{u_i}/a_{u_i} \rightarrow 1.74\ldots$ (roughly 80%) and $b_{l_i}/a_{l_i} \rightarrow 1.408\ldots$ (roughly 20%). (The x-axes in the lower figures are relabeled corresponding to the new sequences.)

FIGURE 4. The two upper figures illustrate the conjectured convergence (Conjecture 1) of the ratio b_n/a_n for games (p, q)GDWN, whenever the pair $(p, q) \notin \mathcal{W}$. Here we view $(2, 4)$GDWN and $(7, 12)$GDWN respectively. The computations include the first ≈ 32000 and ≈ 42000 upper P-positions respectively. In each case, the upper P-beams do not seem to split, rather $b_n/a_n \to 1.618\ldots$. But, clearly the P-positions are perturbed as compared to the corresponding ratio of the coordinates of the upper P-positions of Wythoff Nim, the lower figure.
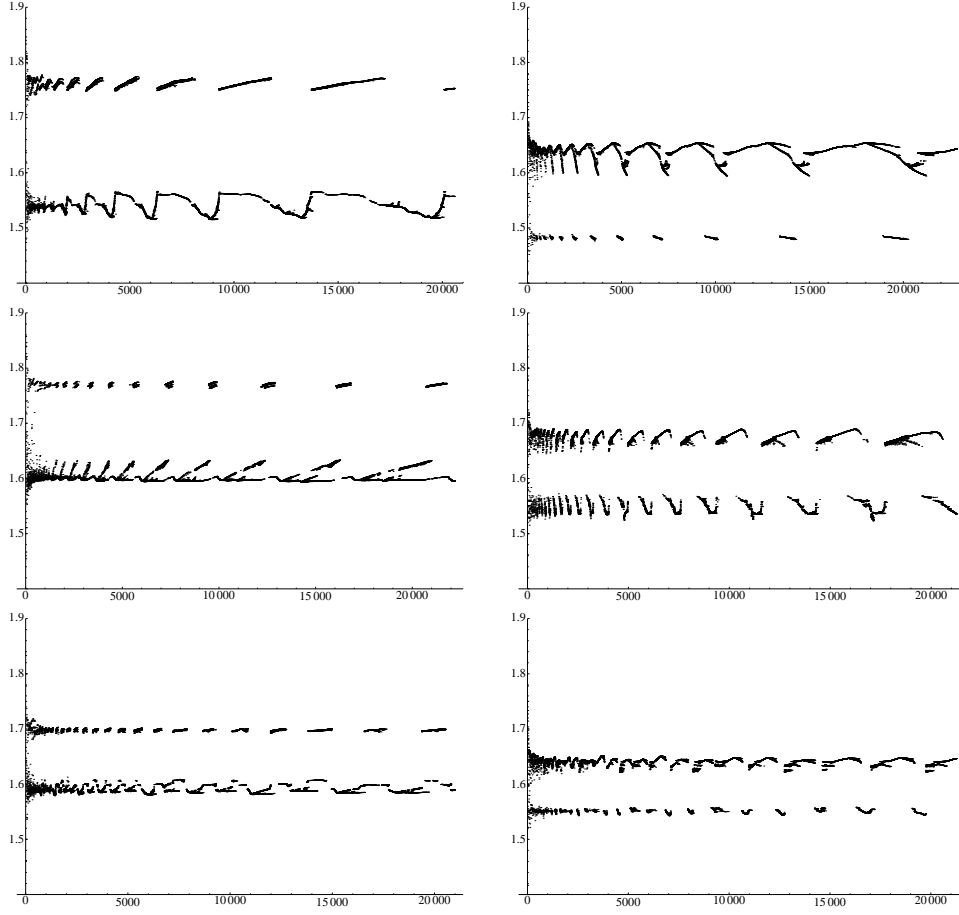
FIGURE 5. The figures illustrate the first $\approx$ 22000 ratios b_n/a_n for $(3,5)$GDWN, $(4,6)$GDWN, $(4,7)$GDWN, $(5,8)$GDWN, $(6,10)$GDWN and $(7,11)$GDWN respectively. Our data seems to suggest that the upper P-positions split for all these games, but it appears that the weaker form of our conjecture is more applicable for this case. For example, for the game $(4,6)$GDWN (top right) there seems to be a pair of complementary sequences u and l such that for large i, $1.60\ldots < b_{u_i}/a_{u_i} < 1.66\ldots$ and the quotient is 'drifting back and forth' in this interval, but possibly $b_{l_i}/a_{l_i} \to 1.48\ldots$ as $i \to \infty$. Also $(4,7)$GDWN seem to split asymptotically, namely to a pair of complementary sequences u and l such that b_{l_i}/a_{l_i} is 'drifting' in the interval $[1.59, 1.63]$ for large i, but possibly $b_{u_i}/a_{u_i} \to 1.77\ldots$. Notice that the three games on the left hand side correspond to Wythoff pairs, whereas those on the right hand side correspond to dual ditto.
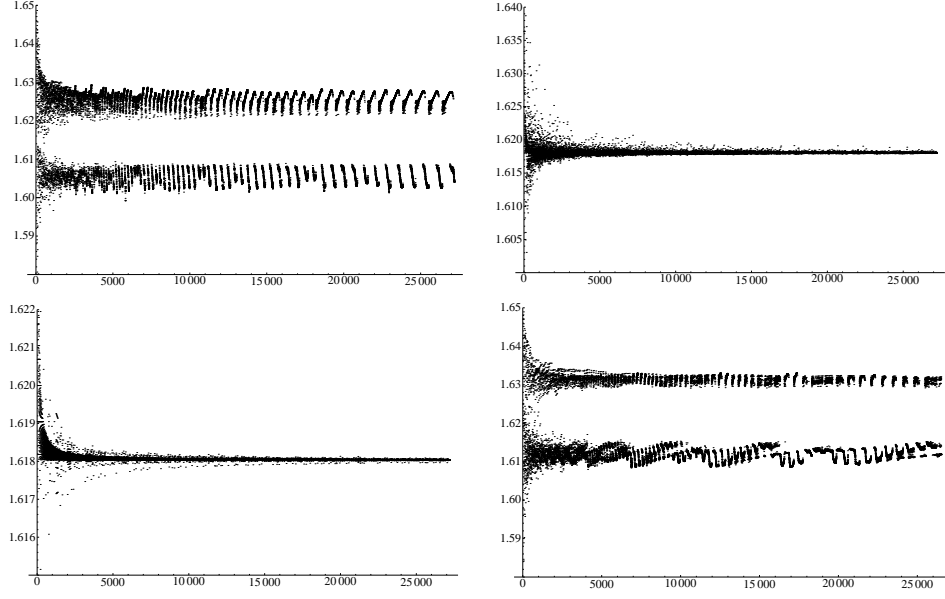
FIGURE 6. These figures illustrate the first ≈ 27000 ratios b_n/a_n for $(31, 50)$GDWN, $(31, 51)$GDWN, $(31, 52)$GDWN and $(32, 52)$GDWN respectively. Notice that $(31, 51)$ and $(31, 52)$ are non-$\mathcal{W}$ pairs, but $52/31 > 1.677 > 51/31 > 1.645 > \phi$. As in Figure 4, there is perturbation of the P-positions of Wythoff Nim for these two games (more accentuated for $(31, 51)$GDWN it appears).
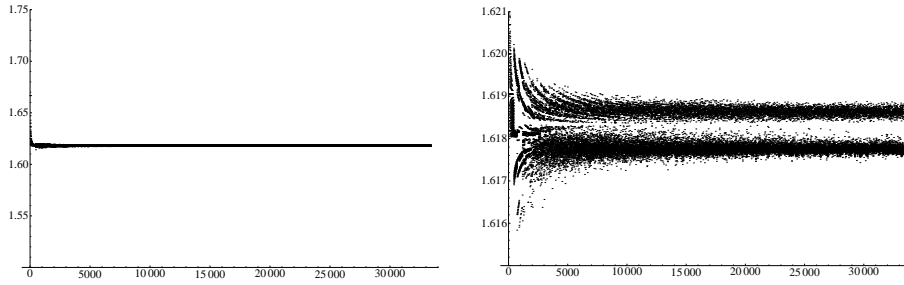

FIGURE 7. The figure illustrates the first ≈ 33000 ratios b_n/a_n for $(731, 1183)$GDWN. The left most figure seems to indicate a convergence to ϕ, but the close-up to the right reveals the conjectured split. Namely $(731, 1183) \in \mathcal{W}$ (a Wythoff pair).

5. Some further experimental results.

At last, we provide some further motivation for a study of (generalizations of) GDWN-games. It is purely experimental. Let SGDWN denote an extension of GDWN, where S is some finite set of (p, q) pairs defining moves of the forms in (1) and where a move is permitted along any of those diagonals. For example, if $S = \{(1, 2), (2, 3)\}$, then all moves of the forms $(t, 2t), (2t, t), (2t, 3t)$ and $(3t, 2t)$ are permitted in addition to the original Wythoff Nim type moves. The P-beams of three such games are given in Figure 8. As a remark for future investigations, it is easy to verify that an analog of Proposition 2.1 holds for these games, that is, the 'new' sequences a and b must be complementary. (For a proof of even further generalizations, see [Lar2]). It is also clear that both Lemma 2.2 and 2.3 continue to hold.
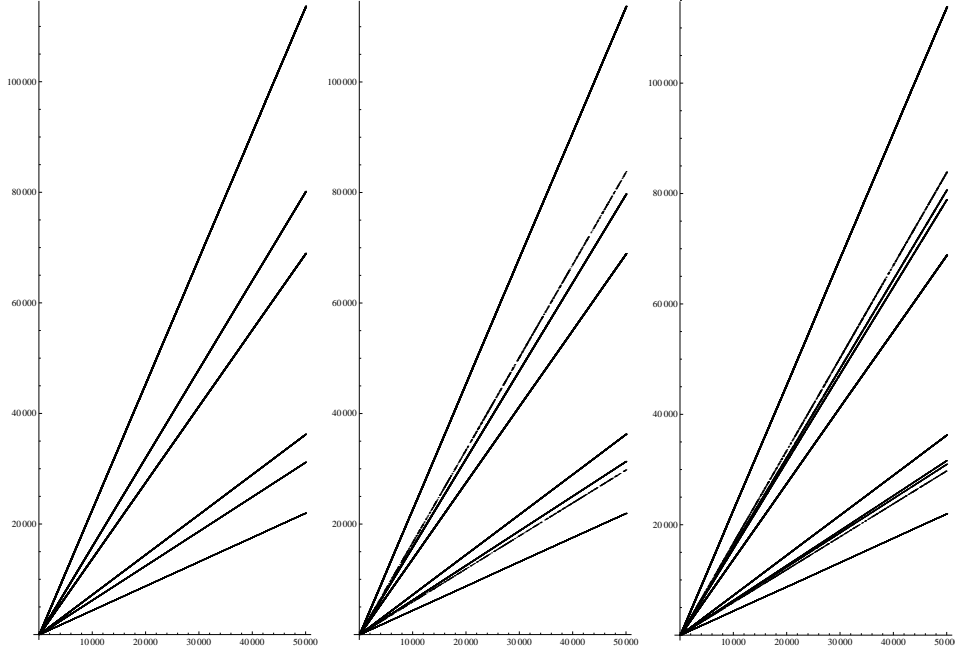

Figure 8. The figures illustrate complete sets of P-positions of $\{(1, 2), (2, 3)\}$GDWN, $\{(1, 2), (2, 3), (3, 5)\}$GDWN and $\{(1, 2), (2, 3), (3, 5), (5, 8)\}$GDWN for all x-coordinates $\leq$ 50000 (which apparently gives y-coordinates $\leq$ 120000). Namely, in Section 5 we generalize the GDWN games and permit moves along several generalized diagonals in one and the same game. A mysterious continuation of the 'splitting of P-beams' from Figure 1 seem to have emerged.

References

[BCG82]   E. R. Berlekamp, J. H. Conway, R.K. Guy, *Winning ways*, **1-2** Academic Press, London (1982). Second edition, **1-4**. A. K. Peters, Wellesley/MA (2001/03/03/04).

[Bou02]   C. L. Bouton, Nim, A Game with a Complete Mathematical Theory *The Annals of Mathematics*, 2nd Ser., Vol. 3, No. 1/4. (1901 - 1902), pp. 35-39.

[Co59]   I.G. Connell, A generalization of Wythoff's game *Can. Math. Bull.* **2** no. 3 (1959), 181-190.

[Con76]   J.H.Conway, *On Numbers and Games*, Academic Press, London (1976). Second edition, A.K.Peters, Wellesley/MA (2001).

[DFNR]   E. Duchêne, A.S. Fraenkel, R.J. Nowakowski and M. Rigo, Extensions and restrictions of Wythoff's game preserving its P-positions. J. Combinat. Theory A 117, 545–567 (2010).

[DFGN]   E. Duchêne, A.S. Fraenkel, S. Gravier and R.J. Nowakowski, Another bridge between Nim and Wythoff, Australasian J. of Combinatorics 44, 43-56, (2009).

[Fra82]   A.S. Fraenkel, How to beat your Wythoff games' opponent on three fronts, *Amer. Math. Monthly* **89** (1982) 353-361.

[FrOz98]   A.S. Fraenkel and M. Ozery, Adjoining to Wythoff's Game its P-positions as Moves. *Theoret. Comp. Sci.* **205**, issue 1-2 (1998) 283-296.

[HeLa06]   P. Hegarty and U. Larsson, Permutations of the natural numbers with prescribed difference multisets, *Integers* **6** (2006), Paper A3, 25pp.

[Lar09]   U. Larsson, 2-pile Nim with a Restricted Number of Move-size Imitations, *Integers* **9** (2009), Paper G4, pp 671-690.

[Lar]   U. Larsson, Restrictions of m-Wythoff Nim and p-complementary Beatty sequences, to appear in *Games of no Chance 4*.

[Lar2]   U. Larsson, Permutation games and the $\star$-operator, preprint.

[LaWä]   U. Larsson, J. Wästlund, Maharaja Nim: Wythoff's Queen meets the Knight, preprint.

[Lar11]   U. Larsson, Blocking Wythoff Nim, *The Electronic Journal of Combinatorics*, P120 of Volume 18(1) (2011).

[Lot01]   M. Lothaire, Algebraic combinatorics on words, Encyplodeia of mathematics and its applications (2001)

[Wyt07]   W.A. Wythoff, A modification of the game of Nim, *Nieuw Arch. Wisk.* **7** (1907) 199-202.

E-mail address: `urban.larsson@chalmers.se`

Mathematical Sciences, Chalmers University Of Technology and Göteborg University, Göteborg, Sweden