Impartial games whose rulesets produce given continued fractions

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Abstract

We study 2-player impartial games of the form *take-away* which produce P-positions (second player winning positions) corresponding to so-called *complementary Beatty sequences*, given by the continued fractions $(1; k, 1, k, 1, \ldots)$ and $(k+1; k, 1, k, 1, \ldots)$. Our problem is the opposite of the main field of research in this area, which is to, given a game, understand its set of P-positions. We are rather given a set of (candidate) P-positions and look for "simple" rules. Our rules satisfy two criteria, they are given by a *closed formula* and they are *invariant*, that is, the available *moves* do not depend on the position played from (for all options with non-negative coordinates).

1 Introduction

This paper assumes the reader is familiar with certain background material, for more information, see the references: For standard terminology of impartial removal games on heaps of tokens, see [WW], for Beatty sequences, see [B], for k-Wythoff Nim, see [W, F], for Sturmian words, see [L], for continued fractions, see [K].

Our problem is an inverse to that of the main field of research, which for a given an impartial ruleset Γ, (for example, $\Gamma = k$-Wythoff Nim) is to determine the P-positions of Γ (within reasonable time-complexity). Here we rather start with a particular (candidate) set of P-positions and search for "simple" game rules. Let us explain the setting.

Throughout this paper, we will denote the *position* consisting of two heaps of $x \geq 0$ and $y \geq 0$ tokens as (x, y). When the values of x and y are known, we adopt the convention that $x \leq y$, though in general we regard such a position as an unordered multiset, so we identify (y, x) with (x, y).

Similarly, we let the *move* (u, v) denote a removal of $v > 0$ tokens from one of the heaps and $0 \leq u \leq v$ from the other, thus from the position (x, y), the move (u, v) is ambiguous, being either $(x, y) \to (x - u, y - v)$, provided both $x - u \geq 0$ and $y - v \geq 0$, or $(x, y) \to (x - v, y - u)$, provided both $x - v \geq 0$ and $y - u \geq 0$. Thus in general, it is necessary to examine both cases.

Recall that a (homogeneous) *Beatty sequence* is a sequence of integers of the form $(\lfloor n\gamma \rfloor)$, the modulus γ being a positive irrational and n ranging over the non-negative integers, here denoted by $\mathbb{N}$. We are interested in positions of the form

$$(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor) \tag{1}$$

for $n \in \mathbb{N}$, where $0 < \alpha < \beta$ are irrationals with

$$\alpha^{-1} + \beta^{-1} = 1, \tag{2}$$

that is $1 < \alpha < 2 < \beta$. By (2), the sequences $(\lfloor n\alpha \rfloor)$ and $(\lfloor n\beta \rfloor)$, where n ranges over the positive integers, $\mathbb{Z}^+$, are *complementary*, (see [B]), that is, each positive integer is attained precisely once in precisely one of these sequences.

In k-Wythoff Nim [F], the P-positions correspond to all unordered pairs of the form in (1) with

$$\alpha = [1; k, k, k, \ldots] = \frac{2 - k + \sqrt{k^2 + 4}}{2}$$

and

$$\beta = [k + 1; k, k, k, \ldots] = \frac{2 + k + \sqrt{k^2 + 4}}{2} = \alpha + k,$$

where $x = [x_1; x_2, x_3, x_4, \ldots]$ denotes the unique continued fraction expansion, CF, of x,

$$x = x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cfrac{1}{x_4 + \cdots}}}.$$

For a variation, in [DR] game rules are examined for P-positions corresponding to the CFs

$$[1; 1, k, 1, k, \ldots] \text{ and } [k+1; k, 1, k, 1, \ldots]. \tag{3}$$

In this paper, we rather study the CF

$$\alpha = \alpha_k = [1; k, 1, k, 1, k, \ldots] = \frac{1 + \sqrt{1 + \frac{4}{k}}}{2}$$

with corresponding

$$\beta = \beta_k = [k+1; 1, k, 1, k, 1, \ldots] = \frac{k + 2 + k\sqrt{1 + \frac{4}{k}}}{2} = k\alpha + 1 = k\alpha^2.$$

Note that $\alpha_k \in (1, 1 + 1/k)$ and $\beta_k \in (k+1, k+2)$. (see Figure 1 and Lemma 12)

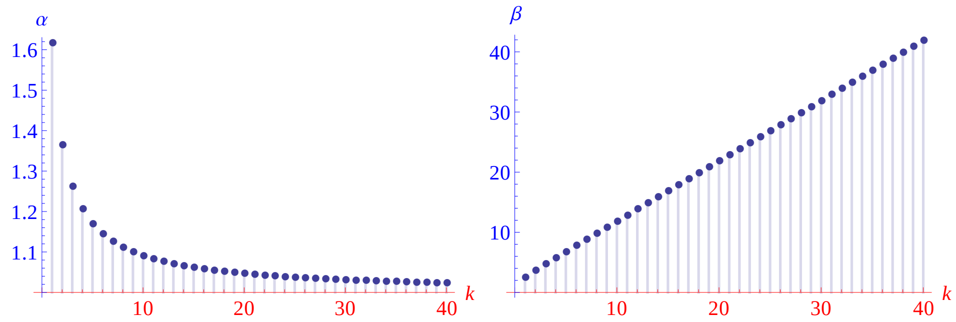

Figure 1: The numbers α_k and β_k for $k \in \{1, 40\}$. See also Figure 5.

Notation 1. For each $k \in \mathbb{Z}^+$, for all $n \in \mathbb{N}$, we let $a_n = \lfloor n\alpha \rfloor$, $b_n = \lfloor n\beta \rfloor$, $c_n = a_n - a_{n-1}$ and $d_n = b_n - b_{n-1}$. Moreover we define the following sequences, $A = (a_1, a_2, \ldots), B = (b_1, b_2, \ldots), C = (c_1, c_2, \ldots)$ and $D = (d_1, d_2, \ldots)$.

Then, for all $n \in \mathbb{Z}^+$,

$$b_n = \sum_{j=1}^{n} d_j \text{ and } a_n = \sum_{j=1}^{n} c_j.$$

Example 2. For $k = 2$,

$$\begin{aligned}
A &= (1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 15, 16, 17, 19, 20, 21, 23, 24, \ldots) \\
C &= (1, 1, 2, 1, 1, 2, 1, 1, 2, 1, 2, 1, 1, 2, 1, 1, 2, 1, \ldots) \\
B &= (3, 7, 11, 14, 18, 22, 26, 29, 33, 37, 41, 44, 48, 52, 55, \ldots) \\
D &= (3, 4, 4, 3, 4, 4, 4, 3, 4, 4, 4, 3, 4, 4, 3, \ldots)
\end{aligned}$$

Since $\alpha_k \in (1, 1 + 1/k)$ and $\beta_k \in (k + 1, k + 2)$, for each k and all n, we also get that

$$c_n \in \{1, 2\} \text{ and } d_n \in \{k + 1, k + 2\}$$

(see [L]). Moreover, each value is attained infinitely often, a statement which we strengthen in Section 2.

Henceforth, for a fixed $k \in \mathbb{Z}^+$, let

$$S_k = \{(a_n, b_n) \mid n \in \mathbb{N}\}.$$

Note that the special case of S_1 corresponds precisely to the P-positions of Wythoff Nim.

The problem of finding a closed formula ruleset such that the set of all P-positions is identical to S_2, was posed by A. S. Fraenkel at the GONC 2011 workshop at the Banff Centre. Here we resolve the general case for the set of (candidate) P-positions being S_k. Henceforth we omit the word "candidate" and simply talk about sets of P-positions. We have also added the requirement that the ruleset be *invariant* [DR, LHF], that is, the available moves do not depend on the position (for all options with non-negative coordinates). This criterion is implicitly fulfilled by many classical removal games, e.g. Nim, k-Wythoff Nim, Subtraction games [WW] and S. Golombs take away-games [Go]. Without the requirement of invariance, one may define the most trivial game rules, no move is possible from a position in S, and otherwise each position has a move to $(0, 0)$. On the other hand, the problem of finding invariant (but not necessarily simple) game rules for any set of P-positions, defined by a complementary pair of homogeneous Beatty sequences, was resolved in [LHF]. However those game rules are not simple in the meaning that the only known formula for the invariant moves is exponentially slow in $\log(xy)$. See Figures 2, 3 and 4 for invariant games corresponding to the CFs on page 3, cases $k = 2$.

For many classical games, such as *normal play* Nim and k-Wythoff Nim, the final winning position is unique, namely $(0, 0)$. Given our set of P-positions, S_k, this requirement clearly needs to be satisfied. A convenient way to achieve this is to follow the example of our classical games, to include the Nim rules to our new game. An immediate benefit of doing this is that we automatically satisfy one of the other inherent requirements of the set S_k, namely that there can be at most one P-position in each row and column of $\mathbb{N} \times \mathbb{N}$. Precisely, the desired ruleset Γ_k has the following permitted moves:

Theorem 3. *Let the set S_k be defined by the Beatty sequences where $\alpha = [1; k, 1, k, 1, k, \ldots]$ and $\beta = [k+1; 1, k, 1, k, 1, \ldots]$, that is $S_k = \{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor) \mid n \in \mathbb{N}\}$. Then the invariant ruleset $\Gamma = \Gamma_k$ consisting of the following moves has a set of P-positions identical to the set S_k (in all cases, $n, s, t \in \mathbb{Z}^+$):*

Type I - Nim Moves

$(x, y) \to (x - n, y)$ or $(x, y) \to (x, y - n)$.

Type II - Extended Diagonal Moves

$(x, y) \to (x - s, y - t)$ provided that $|s - t| < k$ or that $|s - t| = k$ and $s, t > 2$. This is similar to the rules of $(k + 1)$-Wythoff Nim, with the exception that the moves $(s, t) = (1, k + 1)$ and $(2, k + 2)$ are excluded.

Type III - Extra Moves

For $i = 1$ to $k - 1$, use the initial value $(f_0^i, g_0^i) = (0, i + 1)$ and define recursively for $n > 0$,

$$(f_n^i, g_n^i) = (f_{n-1}^i + g_{n-1}^i, kf_{n-1}^i + (k + 1)g_{n-1}^i + i)$$

The extra moves for each i are $(f_n^i, g_n^i - 1)$ for $n > 0$.

Note that when $n = 0$, the move $(0, i)$ is already in the ruleset as it is a Nim move. In Section 2, we will have need to back up the recursion one step and use $(f_{-1}^i, g_{-1}^i) = (-1, 1)$ for all i.

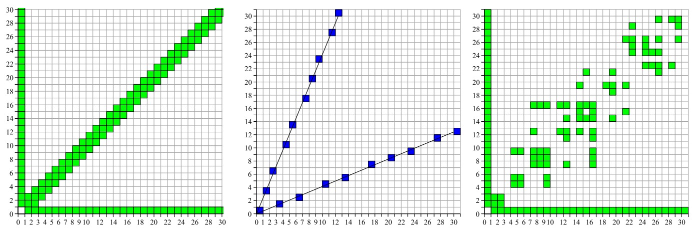

Figure 2: To the left we display the initial moves of the classical game of 2-Wythoff Nim and in the middle its initial P-positions together with the corresponding slopes. To the right we give the initial invariant moves for the game (2-Wythoff Nim)**, with notation as in [LHF], which has P-positions of the same form as those of 2-Wythoff Nim. (The moves are defined via a simple greedy algorithm.)

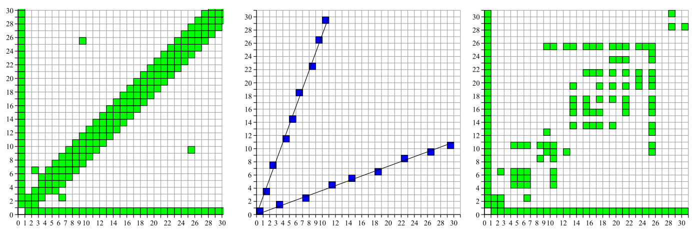

Figure 3: The left most figure displays the initial moves of our game for $k = 2$ as given in Theorem 3 (see also Example 4 for the extra moves). In the middle we see the P-positions and to the right the initial moves of the invariant game from [LHF] with P-positions identical to the set S_2.

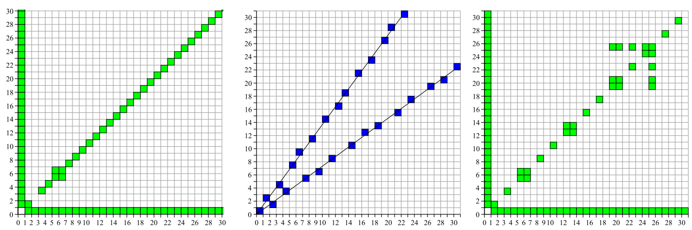

Figure 4: These figures represent the moves (left figure) resolving the P-positions (middle figure) given by the continued fraction (3) with $k = 2$ from [DR]. The right most figure gives the initial moves of the invariant game from [LHF] with identical P-positions.

Example 4. For $k = 2$, (and therefore $i = 1$), the extra moves are

$$(2, 6), (9, 25), (35, 96), \ldots$$

An explicit formula for the Type III moves (f_n^1, g_n^1) is given by

$$f_n^1 = \frac{(1+\sqrt{3})(2+\sqrt{3})^n + (1-\sqrt{3})(2-\sqrt{3})^n - 2}{4}; \tag{4}$$

$$g_n^1 = \frac{(2+\sqrt{3})^{n+1} + (2-\sqrt{3})^{n+1} - 2}{2}. \tag{5}$$

Explicit formulas can be found for larger values of k, but are not as succinct, and therefore we have chosen to omit them.

Example 5. For $k = 4$, the extra moves are

$$i = 1 : (2, 10), (13, 63), (77, 372), \ldots$$
$$i = 2 : (3, 16), (20, 98), (119, 576), \ldots$$
$$i = 3 : (4, 22), (27, 133), (163, 780), \ldots$$

The extra moves are necessary for positions of the form $(a_n, b_n - 1)$ where $c_n = 1$ and $d_n = k + 2$. For instance, when $k = 4$, we seek a winning move from the N-position $(38, 185)$ where $(38, 186)$ is a P-position. The previous P-positions are $(37, 180)$, $(36, 174)$, $(35, 169)$ and $(33, 163)$ with differences from one P-position to its predecessor of $(1, 6)$, $(1, 6)$, $(1, 5)$ and $(2, 6)$ respectively. Preceding the nearest lesser difference of $(2, 6)$ are *two* copies of $(1, 6)$ (ignoring the $(1, 5)$). The winning move uses the largest valid move from the extra move set with $i = 2$, namely the move $(20, 98)$ which moves from $(38, 185)$ to the P-position $(18, 87)$.

The next section develops the machinery to examine these positions and corresponding moves. The final section shows that the rules described in Theorem 3 produce the prescribed set of P-positions.

Remark 6. We are mainly interested in the cases $k \geq 2$ as the set S_1 is the set of P-positions for 1-Wythoff Nim. The moves defined in Theorem 3 for the case $k = 1$ are valid, though different than the moves for 1-Wythoff Nim. Specifically, while Theorem 3 provides no Type III moves in the case $k = 1$, it does have Type II moves which are not included in the set of moves of 1-Wythoff Nim. (A similar observation holds for the case $k = 1$ in [DR].)

2 The Sturmian Word and Morphism Construction of the Beatty Sequence

Here, we lay the groundwork for finding Type III winning moves for positions of the form $(a_n, b_n - 1)$, where $c_n = 1$ and $d_n = k + 2$. We use some terminology from *Sturmian words* and *morphisms* [L]. After some preliminaries, we produce the *characteristic word* which corresponds to the D sequence (this is Lemma 10 which is proved in the Appendix) and thereby gives an alternative description of the B sequence. From it, we find a new characterization of the C and A sequences and note some important properties. Finally, we give an algorithm for finding the desired winning move in Lemma 25.

Lemma 7. *For all* $n \in \mathbb{N}$ *and* $i \in \{1, \ldots, k-1\}$, $g^i_{n+1} = (k+2)g^i_n - g^i_{n-1}$.

Proof: $g^i_{n+1} = (k+1)g^i_n + kf^i_n + i = (k+2)g^i_n - g^i_n + kf^i_n + i = (k+2)g^i_n - (kf^i_{n-1} + (k+1)g^i_{n-1} + i) + k(f^i_{n-1} + g^i_{n-1}) + i = (k+2)g^i_n - g^i_{n-1}$. □

2.1 The sequence $D = (d_1, d_2, d_3, \ldots)$

We wish to describe the sequence $D = (d_1, d_2, d_3, \ldots)$ via the Sturmian word produced by the morphism

$$\varphi(\sigma) = \sigma\tau^k = \sigma\tau\tau\tau\ldots\tau \quad (k \text{ copies of } \tau)$$

$$\varphi(\tau) = \sigma\tau^{k+1} = \sigma\tau\tau\tau\ldots\tau \quad (k+1 \text{ copies of } \tau)$$

and

$$\varphi(uv) = \varphi(u)\varphi(v)$$

for any words u, v consisting of the letters σ, τ where the operation is concatenation.

Notation 8. Let w_0 be the word σ, and $w_n = \varphi(w_{n-1})$. Note that w_{n-1} is a prefix of w_n so that

$$W = \lim_{n\to\infty} \varphi^n(w_0)$$

is well-defined.

Example 9. For $k = 2$, $W = \sigma\tau\tau\sigma\tau\tau\tau\sigma\tau\tau\tau\sigma\tau\tau\sigma\tau\tau\tau\sigma\tau\tau\tau\sigma\tau\tau\tau\sigma\tau\tau\sigma\tau\tau\tau\ldots$

Lemma 10. *For all k, $d_j = k+1$ if the j^{th} letter of W is σ and $d_j = k+2$ if the j^{th} letter of W is τ.*

We give the proof of Lemma 10 in the Appendix.

2.2 The sequence $C = (c_1, c_2, c_3, \ldots)$

Next, we give an equivalent construction of the sequence $C = (c_1, c_2, c_3, \ldots)$ via the D sequence.

Notation 11. The *remainder* or *fractional part* of $x \in \mathbb{R}$ shall be denoted $\{x\} = x - \lfloor x \rfloor$.

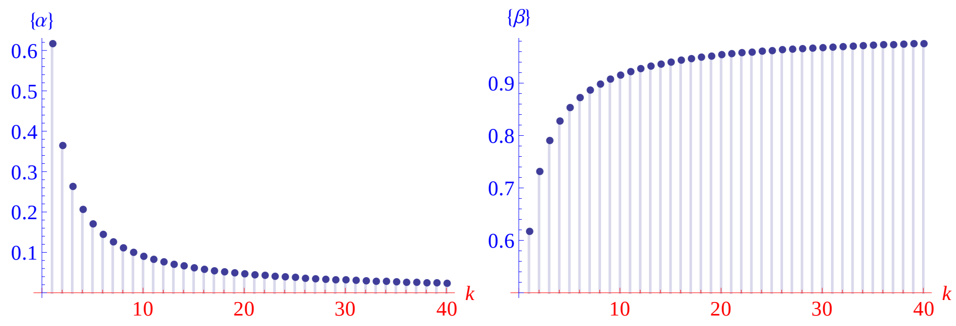

Figure 5: The fractional parts $\{\alpha_k\}$ and $\{\beta_k\}$, for $k \in \{1, 40\}$.

Lemma 12. *For all k, $\{\alpha_k\} < 1/k$.*

Proof: $1 < \alpha < 2$, so $\{\alpha\} = \alpha - 1 = \frac{1}{2}\left(\sqrt{1+\frac{4}{k}} - 1\right)$.

$$\text{Since } (k+2)^2 > k^2\left(1+\frac{4}{k}\right)$$

$$\implies \frac{k+2}{k} > \sqrt{1+\frac{4}{k}}$$

$$\implies \frac{1}{k} > \frac{1}{2}\left(\sqrt{1+\frac{4}{k}} - 1\right) = \{\alpha\}.$$

□

Lemma 13. *For all k, $\{\beta_k\} = k\{\alpha_k\}$.*

Proof:

$$\beta = k\alpha + 1 = k + 1 + k\{\alpha\}$$

$$\implies \{\beta\} = k\{\alpha\}$$

since $\{\alpha\} < \frac{1}{k} \implies k\{\alpha\} < 1$. $\square$

We now compare the the sequences C and D, first with an example

Example 14. For $k = 4$,

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1\,1\,1\,1\,2\,1\,1\,1\,1\,2\,1\,1\,1\,1\,2\,1\,1\,1\,1\,2\,1\,1\,1\,1\,2\,1\,1\,1\,2\,1\,1\,1\,1\,2\,1 & \cdots \\ 5\,6\,6\,6\,6\,5\,6\,6\,6\,6\,6\,5\,6\,6\,6\,6\,6\,5\,6\,6\,6\,6\,6\,5\,6\,6\,6\,6\,6\,5\,6\,6\,6\,6\,5 & \end{pmatrix}$$

If we remove each 5 in the D sequence and the corresponding 1 in the C sequence, what remains in the C sequence is periodic with the value 2 in positions $4, 8, 12, \dots$ and the value 1 otherwise. It turns out that this observation corresponds to an alternative description of the C sequence provided by the D sequence, for general k.

Lemma 15. *Suppose that $c_p = c_q = 2$ for $p > 0$ and $q > p$ minimal. Then there are exactly $k - 1$ values of i, $p < i < q$ for which $d_i = k + 2$.*

Proof: Since $c_p = a_p - a_{p-1} = 2$, Lemma 12 gives $\{p\alpha\} < \frac{1}{k}$. Let $i \in \{p + 1, \dots, q - 1\}$ so that $c_i = 1$ and so (the latter inequality is by Lemma 12),

$$0 < \{\alpha\} = \{i\alpha\} - \{(i - 1)\alpha\} < \frac{1}{k}. \tag{6}$$

By Lemma 13, this gives $\{\beta\} = k\{i\alpha\} - k\{(i - 1)\alpha\}$. Now, as we have seen, going from b_{i-1} to b_i produces either the difference $d_i = k+1$ or $k+2$. Then, by $\beta \in (k + 1, k + 2)$, it is clear that the greater value will be attained if and only if there is a $j \in \{1, \dots, k - 1\}$ such that $\{(i - 1)\alpha\} < \frac{j}{k} < \{i\alpha\}$.

By the last inequality in (6), each j will correspond to a unique i. Hence $d_i = k + 2$ occurs exactly $k - 1$ times between consecutive occurrences of $c_n = 2$. $\square$

Lemma 16. *Let $k \in \mathbb{Z}^+$. Then $c_n = 2$ iff $d_n = k + 2$ and $b_n \equiv n$ (mod k).*

Proof: Suppose now that $d_i = k + 1$ so that

$$0 < \{\beta\} = \{i\beta\} - \{(i-1)\beta\} < 1. \tag{7}$$

Then $0 < k\{\alpha\} = \{ik\alpha\} - \{(i-1)k\alpha\} < 1$. If in addition $c_i = 2$ we get that $0 < \{ik\alpha\} = k\{i\alpha\} < 1$, so that $0 < \{i\alpha\} - \{\alpha\} = \{(i-1)k\alpha\}/k < 1/k$. This gives that $\{i\alpha\} > 1/k$ so that $c_i = 1$. Thus, we have proved that $c_n = 2$ implies $d_n = k+2$. But then Lemma 15 gives that $b_n - (k+1)n \equiv 0 \pmod{k}$, for each n such that $c_n = 2$. □

For record keeping purposes, we index the τ in the word W with period k so that

$$W = \sigma\tau_1 \ldots \tau_k\sigma\tau_1 \ldots \tau_k\tau_1\sigma\tau_2 \ldots$$

Definition 17. A *syllable* of W is a string of letters of the form $\varphi(\sigma)$ or $\varphi(\tau)$, that is, it begins with σ, and ends with the τ which precedes the next σ.

Thus the morphism φ maps letters to syllables. Note that the indexing for the $\varphi(\sigma)$ will depend on the preceding syllable, but each index will appear exactly once. Hence, for all i, we get that

$$\varphi(\tau_i) = \sigma\tau_i\tau_{i+1} \ldots \tau_k\tau_1 \ldots \tau_i.$$

Using this notation Lemma 16 states that

$$c_n = 2 \text{ iff the } n^{th} \text{ letter of } W \text{ is } \tau_k. \tag{8}$$

2.3 Sums of Factors

Definition 18. A *factor* of a word is a sequence of consecutive letters. If the factor begins with the first letter of the word, the factor is called a *prefix.* If the factor contains the last letter of a finite word, the factor is called a *suffix.*

Definition 19. For each $i \in \{1, \ldots, k-1\}$, let w_0^i be the word $\sigma\tau_1 \ldots \tau_i$ and $w_n^i = \varphi(w_{n-1}^i)$.

Lemma 20. *For each $i \in \{1, \ldots, k-1\}$ and all $n \geq 0$, f_n^i counts the number of copies of τ_k in the word w_n^i and g_n^i counts the number of letters. Note that for $n \geq 1$, g_{n-1}^i counts the number of syllables in the word (which equals the number of copies of σ in w_n^i by construction).*

Proof: Base Case: $f_0^i = 0$ and w_0^i contains no τ_k. w_0^i contains $i+1$ letters and $g_0^i = i+1$.

Induction: The morphism φ sends each τ_k to a syllable containing two τ_k and all other letters to a syllable containing a single τ_k, hence the number of copies of τ_k in w_{n+1}^i is $2f_n^i + (g_n^i - f_n^i) = f_{n+1}^i$. The number of letters in the new word is $k+2$ for each letter subtracting one for each σ for a total of $(k+2)g_n^i - g_{n-1}^i = g_{n+1}^i$ by Lemma 7. □

Lemma 21. *A factor of W of length g_n^i contains either g_{n-1}^i or $g_{n-1}^i - 1$ copies of σ. No other number is attainable.*

Proof: By construction, w_n^i has length g_n^i and has g_{n-1}^i copies of σ so g_{n-1}^i is attainable. W is a Sturmian word, and therefore balanced, hence only one other value is attainable, either $g_{n-1}^i - 1$ or $g_{n-1}^i + 1$. Shift $k+2$ steps to the right in w_n^i. Then we lose 2 copies of σ and gain one. Hence $g_{n-1}^i - 1$ is the correct value. □

Lemma 22. *For each $i \in \{1, \ldots, k-1\}$, and for all $n \geq 1$, (f_n^i, g_n^i) is a P-position.*

Proof: Let $j = g_{n-1}^i$, which is the length of w_{n-1}^i by Lemma 20. By (8), the number of copies of τ_k plus the number of letters in w_{n-1}^i is a_j. By Lemma 20 this equals $f_{n-1}^i + g_{n-1}^i = f_n^i$. By construction and Lemma 7, $b_j = (k+2)j -$ (the number of copies of σ in w_{n-1}^i) $= g_n^i$. □

Definition 23. A factor of W has *index i* if it ends with τ_i for some $i \in \{1, \ldots, k-1\}$. A P-position (a_n, b_n) has *index i* if the prefix of W of length n has index i.

Lemma 24. *For a fixed index i, let x be a factor of the word w_{n+2}^i, with the following properties:*

- *x has length g_{n+1}^i*
- *x is not the suffix of w_{n+2}^i*

- *x ends in τ_i*

Then x contains precisely g_n^i copies of σ. By construction, two equal length factors of W with the same index and the same number of copies of σ will correspond to two equal length factors of C with the same number of copies of 2. Hence the two factor sums in C are equal and the two factor sums in D are equal.

Proof: Note that the statement is vacuous if $n = -2$

Base Case: $n = -1$

If $n = -1$, then x has length $i + 1$. w_1^i has $i + 1$ syllables, with τ_i in position $i + 1$ in the first syllable and in position $i + 3 - s$ in syllable s for $2 \leq s \leq i + 1$, hence the $i + 1$ letters ending in τ_i always contain exactly one σ.

Induction:

If the terminal τ_i of the factor x is the last letter of a non-terminal syllable of w_{n+2}^i, then the factor contains exactly g_n^i syllables since the terminal τ_i was a result of the output of $\varphi(\tau_i)$, and the previous word w_{n+1}^i has precisely g_{n-1}^i copies of σ in a factor of length g_n^i by induction.

If the terminal τ_i is not the last letter in its syllable, then compare the factor x with the nearest previous factor y for which the terminal τ_i is the last letter in its syllable. If the factors x and y overlap so that there exist non-empty words t, u, v with $y = tu$, $x = uv$, we need to show that the number of copies of σ in t equals the number of copies of σ in v.

Let j be the index of the syllable containing the terminal τ_i of the factor x. If there is no syllable $\varphi(\sigma)$ in v, then the length of v is $(k + 1)m$ where $m = j - i$ if $j > i$ and $m = k + j - i$ if $j \leq i$. v contains $m - 1$ full syllables plus the terminal partial syllable. Each full and partial syllable contains one σ, so v contains m copies of σ. In other words, the fraction of letters in v which are σ is $\dfrac{1}{k+1}$. If v does contain a syllable $\varphi(\sigma)$, this ratio is unchanged.

For any integer m, the number of copies of σ in any factor of length $(k+1)m$ cannot exceed this ratio since the length of each syllable is $\geq k + 1$. Since the number of copies of σ in y is g_n^i which is maximal by Lemma 21, the number of copies of σ in x which is at least as many as in y must also be g_n^i hence the number of copies of σ in y equals the number of copies of σ in x.

In the case that x and y do not overlap, note that the maximum distance that x needs to be shifted occurs when the terminal τ_i of x is the leading τ in a syllable ending in τ_i and that this distance is $k-1$ syllables of length $k+2$ plus perhaps a syllable of length $k+1$ plus 2 for a total of $[(k-1)(k+2)]+[k+1]+2 = (k+1)^2 < g_2^i$ thus in the induction step, x and y do not overlap only for $n = 0$. In this case, y has one syllable of length $k+1$ and i syllables of length $k+2$ yielding $i+1$ copies of σ. x has its terminal partial syllable of length $\leq k+1$, thus x has at least as many copies of σ as does y, and since the number of copies of σ in y is maximal by Lemma 21, the number of copies of σ is the same in x and y. □

At the beginning of this section we promised an algorithm for finding a certain winning move. We deliver it here:

Lemma 25. *Let $(x, y) = (a_n, b_n)$ be a P-position with index $i \in \{1, \ldots, k-1\}$. From the position $(x, y-1)$, the Type III move (u, v) corresponding to i with $v \leq b_n - 1$ maximal is to a P-position.*

Proof: Find m such that $g_m^i \leq b_n < g_{m+1}^i$. In the first case, if $b_n = g_m^i$, then from $(x, y-1)$, the extra move $(f_m^i, g_m^i - 1)$ is to $(0, 0)$ by Lemma 22. In all other cases, Lemma 24 shows that all factors with index i and length g_{m-1}^i, except the last, in the word w_m^i have the same number of copies of σ, hence the factor sums in C and D have the same sums as in the first case, namely f_m^i and $g_m^i - 1$. Hence the move $(f_m^i, g_m^i - 1)$ is to the P-position (a_{n-j}, b_{n-j}) where $j = g_{m-1}^i$. □

3 The rules are correctly defined

In this section, we prove Theorem 3, that is we verify that the set S_k is generated as the complete set of P-positions by the ruleset Γ_k.

Definition 26. The *gap* of a P-position (a_n, b_n), denoted δ_n is

$$\delta_n = b_n - a_n$$

The *gap difference* between two P-positions (a_m, b_m) and (a_n, b_n) with $m > n$, denoted $\Delta(m, n)$ is

$$\Delta(m, n) = \delta_m - \delta_n$$

We must check that there is no move connecting any two P-positions (such a "short-circuit" would force at least one of the P-positions to be de facto N and so we had to exclude it from the set S_k) and that every N-position has a move to a position in the set S_k (for otherwise one of the N-positions would be de facto P, and so we had to include it to the set S_k).

Proof - Part I - No move connects P to P:

By the complementarity of the Beatty sequences, moves of Type I cannot connect any two P-positions.

Note that $\Delta(m, m-1) = k$ or $\Delta(m, m-1) = k+1$. Recall that $a_m - a_{m-1} \leq 2$, so moves of Type II cannot connect (a_m, b_m) and (a_{m-1}, b_{m-1}). If $m-n > 1$, then $\Delta(m, n) > k$, so moves of Type II cannot connect P-positions in this case either.

It remains to justify that moves of Type III never connect two P-positions. Let (p, q), $p < q$ be an extra move so that $(p, q+1) = (a_i, b_i)$, for some positive integer i, by Lemma 22. From the P-position (a_m, b_m), it is clear that $(a_m - q, b_m - p)$ is not a P-position since $(b_m - p) - (a_m - q) > \delta_m$ and the gap must decrease.

To show that $(a_n - p, b_n - q)$ is not a P-position, assume the contrary, and note that

$$\begin{aligned}(a_n - p, b_n - q) &= (a_n - a_i, b_n - b_i + 1) \\ &= (\lfloor n\alpha \rfloor - \lfloor i\alpha \rfloor, \lfloor n\beta \rfloor - \lfloor i\beta \rfloor + 1).\end{aligned} \tag{9}$$

We have

$$\begin{aligned}\lfloor (n-i)\alpha \rfloor &= (n-i)\alpha - \{(n-i)\alpha\} \\ &= n\alpha - i\alpha - \{(n-i)\alpha\} \\ &= \lfloor n\alpha \rfloor + \{n\alpha\} - \lfloor i\alpha \rfloor - \{i\alpha\} - \{(n-i)\alpha\},\end{aligned}$$

but then

$$\lfloor (n-i)\alpha \rfloor - \lfloor n\alpha \rfloor + \lfloor i\alpha \rfloor = \{n\alpha\} - \{i\alpha\} - \{(n-i)\alpha\} \tag{10}$$

must be an integer and is therefore either 0 (if $\{n\alpha\} \geq \{i\alpha\} + \{(n-i)\alpha\}$) or -1 (if $\{n\alpha\} < \{i\alpha\} + \{(n-i)\alpha\}$).

CASE 1: $\{n\alpha\} - \{i\alpha\} - \{(n-i)\alpha\} = 0$. Then, (9) and (10) give that

$$\begin{aligned} a_n - p &= \lfloor n\alpha \rfloor - \lfloor i\alpha \rfloor \\ &= \lfloor (n-i)\alpha \rfloor \\ &= a_{n-i}. \end{aligned}$$

Hence, for $(a_n - p, b_n - q)$ to be a P-position, we must have

$$\begin{aligned} b_n - b_i + 1 &= b_{n-i} \\ &= \lfloor (n-i)\beta \rfloor \\ &= \lfloor b_n + \{n\beta\} - b_i - \{i\beta\} \rfloor \\ &= b_n - b_i + \lfloor \{n\beta\} - \{i\beta\} \rfloor, \end{aligned}$$

but $\lfloor \{n\beta\} - \{i\beta\} \rfloor$ cannot be 1.

CASE 2: $\{n\alpha\} - \{i\alpha\} - \{(n-i)\alpha\} = -1$. Then (10) gives that

$$\lfloor n\alpha \rfloor - \lfloor i\alpha \rfloor = \lfloor (n-i)\alpha \rfloor + 1.$$

By the latter expression, this number, which is strictly greater than zero, can belong either to the set A or B. If

$$\lfloor (n-i)\alpha \rfloor + 1 = b_x \in B,$$

then, for

$$(a_n - p, b_n - q) = (\lfloor n\alpha \rfloor - \lfloor i\alpha \rfloor, \lfloor n\beta \rfloor - \lfloor i\beta \rfloor + 1)$$

to be a non-trivial P-position, we must have that $a_x = b_n - q$ and $b_x = a_n - p$. But, since for all $x > 0$, $b_x > a_x$, this gives

$$a_n - a_i > b_n - b_i + 1,$$

which is false, since $\delta_n > \delta_i$ if $n > i$.

Otherwise

$$\lfloor n\alpha \rfloor - \lfloor i\alpha \rfloor = a_{n-i+1} \in A,$$

so that, by (9) and the definition of a P-position, we must have

$$\begin{aligned} b_n - b_i + 1 &= b_{n-i+1} \\ &= \lfloor (n-i+1)\beta \rfloor. \end{aligned} \tag{11}$$

However

$$\begin{aligned}\lfloor (n-i+1)\beta \rfloor &= \lfloor b_n + \{n\beta\} - b_i - \{i\beta\} + b_1 - \{\beta\} \rfloor \\ &= b_n - b_i + b_1 + \lfloor \{n\beta\} - \{b_i\} - \{b_1\} \rfloor. \end{aligned} \tag{12}$$

The last term is either 0 or -1. There are no moves of Type III for the case $k = 1$, thus $k \geq 2$ and $\beta > 3$. Therefore $b_1 \geq 3$, which gives $b_1 + \lfloor \{n\beta\} - \{i\beta\} - \{\beta\} \rfloor \neq 1$, which, by (12), contradicts (11).

Proof - Part II - Every N has a move to a P:

Assume in all cases that $x \leq y$.

If (x, y) is an N-position and either $x \in B$ or $y \in B$, then there is a Nim move (Type I) to a P-position. If $x = a_n \in A, y \in A$, with $y > b_n$, the Nim move lowering y to b_n is to a P-position.

If $x = a_n, y \in A, y < b_n - 1$, then $y - x \leq \delta_n - 2$. Since the gaps δ_j increase by either k or $k+1$ as j increases by 1, then there is an Extended Diagonal move (Type II) to a P-position corresponding to δ_j which is nearest $y - x$.

What remains to be shown are winning moves from $x = a_n, y = b_n - 1$. If $a_n = a_{n-1} + 2$ or $b_n = b_{n-1} + k + 1$, then the extended diagonal move $(2, k+1)$ or $(1, k)$ moves to the P-position (a_{n-1}, b_{n-1}). Otherwise, Lemma 25 finds the winning Type III move. □

Appendix

All notation in the appendix is local unless stated otherwise. We use theory from [Gl] (further references are given in [Gl]). The words are defined on the alphabet $\{0, 1\}$. For $k \geq 2$ an integer, we are interested in the morphism

$$\theta: \ 0 \to 1^k 0 \tag{A.1}$$

$$1 \to 1^k 01 \tag{A.2}$$

which we will show corresponds to the positive root,

$$\gamma = \frac{\sqrt{k^2 + 4k} - k}{2} \in (1/2, 1), \tag{A.3}$$

of $x^2 + kx - k = 0$. Namely, by a result in [Gl], we will obtain that

$$\lim_{n \to \infty} \theta^n(1)$$

is the characteristic word c_γ of γ. The density of (the 1s in) c_γ is of course γ. Also β as defined in Section 1 equals $\gamma + k + 1$ (that is $\gamma = \{\beta\}$). Hence the continued fraction expansion of γ is $\gamma = [0; 1, k, \overline{1, k}]$ (where $\overline{x}$ denotes the periodic pattern $x, x, \ldots$).

Let $X = \lim_{n\to\infty} \theta^n(1)$. We show by induction that the Sturmian word $W = \lim_{\to\infty} \varphi(\sigma)$, defined as in Section 2, is identical (via $\sigma \leftrightarrow 0, \tau \leftrightarrow 1$) to the word $0X$. That is, we want to show that:

Lemma A.1. *The i^{th} letter of $0X$ is a 1 if and only if the i^{th} letter of W is a τ.*

Proof: The first letter in $0X$ and W is 0 and σ respectively; φ acts on its letter, whereas θ does not. Rather, θ acts on the first letter in X. Let us state our induction hypothesis:

Case 1, $x_j = 0$: Then the last letter of the j^{th} syllable of θ, as in the right hand side of (A.1), corresponds precisely to the first letter of the $(j+1)^{st}$ syllable of φ.

Case 2, $x_j = 1$: Then the last two letters of the j^{th} syllable of θ, as in the right hand side of (A.2), correspond precisely to the first two letters of the $(j+1)^{st}$ syllable of φ.

If these two cases hold for all j, then, by

$$\text{(the lenght of } \varphi(\sigma)) = (\text{the length of } \theta(0)) = k+1$$

and

$$\text{(the length of } \varphi(\tau)) = (\text{the length of } \theta(1)) = k+2,$$

the infinite words correspond precisely. This follows since then the first k letters in the j^{th} syllable of X, each a copy of 1, correspond precisely to the last k letters of the j^{th} syllable of W, each a copy of τ.

Our base case is that the first syllable of X ends with 01 and the second syllable of W begins with $\sigma\tau$, and indeed it holds for the prefixes 01^k01 and $\sigma\tau^k\sigma\tau^{k+1}$ respectively.

But then, comparing the definitions of φ and θ with the paragraph after Case 2, the induction hypothesis gives the claim. □

Define on $\{0,1\}$ the following three morphisms

$$E: \begin{array}{ccc} 0 & \mapsto & 1 \\ 1 & \mapsto & 0 \end{array}, \qquad \eta: \begin{array}{ccc} 0 & \mapsto & 01 \\ 1 & \mapsto & 0 \end{array}, \qquad \overline{\eta}: \begin{array}{ccc} 0 & \mapsto & 10 \\ 1 & \mapsto & 0 \end{array}.$$

A morphism ψ is *Sturmian* if and only if it is a composition of E, η, and $\overline{\eta}$ in any number and order. Furthermore, a morphism ψ is *standard* if and only if it is a composition of E and η in some order. A morphism is *non-trivial* if it is neither E nor the identity morphism.

Suppose $\alpha = [0; 1+d_1, d_2, d_3, \ldots]$, with $d_1 \geq 0$ and all other $d_n > 0$. To the *directive sequence* $(d_1, d_2, d_3, \ldots)$, we associate a sequence $(s_n)_{n\geq -1}$ of words defined by

$$s_{-1} = 1,\ s_0 = 0,\ s_n = s_{n-1}^{d_n} s_{n-2}; \quad n \geq 1.$$

Such a sequence of words is called a *standard sequence.*

For any $n \geq 0$, s_n is a prefix of s_{n+1}, so that $\lim\limits_{n\to\infty} s_n$ is well defined as an infinite word. Moreover, standard sequences are related to characteristic Sturmian words. Each s_n is a prefix of c_α, and we have

$$c_\alpha = \lim_{n\to\infty} s_n.$$

In [Gl], all irrationals $\alpha \in (0,1)$ such that the characteristic Sturmian word c_α is generated by a morphism are classified. A *Sturm number* is an irrational number $\alpha \in (0,1)$ that has a continued fraction expansion of one of the following types:

(i) $\alpha = [0; 1+d_1, \overline{d_2, \ldots, d_n}] < \frac{1}{2}$ with $d_n \geq d_1 \geq 1$;

(ii) $\alpha = [0; 1, d_1, \overline{d_2, \ldots, d_n}] > \frac{1}{2}$ with $d_n \geq d_1$.

Observe that if $\alpha = [0; 1+d_1, \overline{d_2, \ldots, d_n}]$ with $d_n \geq d_1 \geq 1$, then

$$1 - \alpha = [0; 1, d_1, \overline{d_2, \ldots, d_n}].$$

Hence, α has an expansion of type (i) if and only if $1-\alpha$ has an expansion of type (ii). Accordingly, α is a Sturm number if and only if $1-\alpha$ is a Sturm number and one can show that $c_{1-\alpha}$ is obtained from c_α by exchanging all letters 0 and 1 in c_α, so that

$$c_{1-\alpha} = E(c_\alpha). \tag{A.4}$$

Therefore, we can restrict our attention to characteristic Sturmian words c_α such that α is a Sturm number of type (i).

We say that a morphism ψ *fixes* an infinite word x if $\psi(x) = x$, in which case x is called a *fixed point* of ψ. The following result describes all irrationals $\alpha \in (0,1)$ such that c_α is a fixed point of a non-trivial morphism.

Theorem A.2. *[BS] Let $\alpha \in (0,1)$ be irrational. Then c_α is a fixed point of a non-trivial morphism σ if and only if α is a Sturm number. In particular, if $\alpha = [0; 1 + d_1, \overline{d_2, \ldots, d_n}]$ with $d_n \geq d_1 \geq 1$, then c_α is the fixed point of any power of the morphism*

$$\sigma : \begin{array}{ccc} 0 & \mapsto & s_{n-1} \\ 1 & \mapsto & s_{n-1}^{d_n - d_1} s_{n-2} \end{array} .$$

Our irrational $\gamma = [0; 1, k, \overline{1, k}]$ (as in (A.3)) is of type (ii) for all k, with $n = 3$, $d_1 = k, d_2 = 1, d_3 = k$, and so we rather apply the theorem to $\alpha = 1 - \gamma = [0; 1 + k, \overline{1, k}]$ which is of type (i). For our application, we have that $s_{-1} = 1, s_0 = 0, s_1 = 0^k 1$ and $s_2 = 0^k 10$, so that the morphism σ in Theorem A.2 corresponds to $0 \to 0^k 10$ and $1 \to 0^k 1$. By (A.4), it is easy to check that the standard morphism $(E\eta)^k \eta$ is identical to $E\sigma = \theta$, so that $E(\lim_{n\to\infty} \sigma^n(0))$ corresponds to the characteristic word c_γ with γ as in (A.3). This proves Lemma 10. □

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