BLOCKING WYTHOFF NIM

URBAN LARSSON

Abstract. The 2-player impartial game of Wythoff Nim is played on two piles of tokens. A move consists in removing any number of tokens from precisely one of the piles or the same number of tokens from both piles. The winner is the player who removes the last token. We study this game with a blocking maneuver, that is, for each move, before the next player moves the previous player may declare at most a predetermined number, $k-1 \geq 0$, of the options as forbidden. When the next player has moved, any blocking maneuver is forgotten and does not have any further impact on the game. We resolve the winning strategy of this game for $k = 2$ and $k = 3$ and, supported by computer simulations, state conjectures of the asymptotic 'behavior' of the P-positions for the respective games when $4 \leq k \leq 20$.

1. Introduction

In this note we study a variation of the 2-player combinatorial game of Wythoff Nim [Wyt07]. The game is *impartial*, because given a position in the game, the set of options does not depend on which player is in turn to move. A background on impartial games may be found in [ANW07, BCG82, Con76]. Let $\mathbb{N}$ and $\mathbb{N}_0$ denote the positive and non-negative integers respectively. Let the *game board* be $\mathcal{B} := \mathbb{N}_0 \times \mathbb{N}_0$.

Definition 1. *Let $(x, y) \in \mathcal{B}$. Then $(x-i, y-j)$ is an option of Wythoff Nim if either:*

(v) $0 = i < j \leq y$,
(h) $0 = j < i \leq x$,
(d) $0 < i = j \leq \min\{x, y\}$,

$i, j \in \mathbb{N}$.

In this definition one might want to think about (v), (h) and (d) as symbolizing the 'vertical' $(0, i)$, 'horizontal' $(i, 0)$ and 'diagonal' (i, i) *moves* respectively. Two players take turns in moving according to these rules. The player who moves to the position $(0, 0)$ is declared the winner. Here we study a variation of Wythoff Nim with a *blocking maneuver* [SmSt02].

Notation 1. *The player in turn to move is* the next player *and the other player is* the previous player.

Date: October 27, 2010.
Key words and phrases. Beatty sequence, Blocking maneuver, exact k-covers, Impartial game, Muller Twist, Wythoff Nim.

Definition 2. *Let $k \in \mathbb{N}$. In the game of* Blocking-k Wythoff Nim*, denoted by* W^k*, the options are defined as in Wythoff Nim, Definition 1. But before the next player moves, the previous player may declare at most $k-1$ of them as forbidden. When the next player has moved, any blocking maneuver is forgotten and has no further impact on the game.*

Notice that for $k = 1$ this game is Wythoff Nim. A player who is unable to move, because all options are forbidden, loses.

We adapt the standard terminology of P- and N-positions—the previous and next player winning positions respectively—of non-blocking impartial games to 'k-blocking' ditto.

Definition 3. *The value of (a position of) W^k is P if (strictly) fewer than k of its options are P, otherwise it is N. Denote by $\mathcal{P}_k$ the set of P-positions of* W^k.

By this definition, the next player wins if and only if the position is N. It leads to a recursive definition of the set of P-positions of W^k, see also Proposition 1.2 on page 4. Since both the Wythoff Nim type moves and the blocking maneuvers are 'symmetric' on the game board it follows that the sets of P- an N-positions are also 'symmetric'. Hence, the following notation.

Notation 2. *We use the 'symmetric' notation $\{x, y\}$ for unordered pairs of integers, that is whenever (x, y) and (y, x) are considered the same.*

Let us explain the main results of this paper, see also Figure 1.

Definition 4. *Let $\phi = \frac{1+\sqrt{5}}{2}$ denote the Golden ratio. Then*

$$\mathcal{R}_1 := \{\{\lfloor \phi n \rfloor, \lfloor \phi^2 n \rfloor\} \mid n \in \mathbb{N}_0\},$$
$$\mathcal{R}_2 := \{(0,0)\} \cup \{\{n, 2n+1\} \mid n \in \mathbb{N}_0\} \cup \{(2x+2, 2y+2) \mid (x, y) \in \mathcal{R}_1\},$$

and

$$\mathcal{R}_3 := \{(0,0)\} \cup \{\{n, 2n+1\}, \{n, 2n+2\} \mid n \in \mathbb{N}_0\}\}.$$

Theorem 1.1. *The sets $\mathcal{P}_i = \mathcal{R}_i$, $i = 1, 2, 3$.*

It is well known that the set $\mathcal{P}_1 = \mathcal{R}_1$ [Wyt07]. We prove the latter two results in Section 2. Admittedly, it surprised me that the solutions of these two problems were so pliable.

In Section 3 we give a table of a conjectured asymptotic 'behavior' of $\mathcal{P}_k$ for each $k \in \{4, 5, \ldots, 20\}$ and also give a brief discussion on a certain family of 'Comply games'—in particular we define the game W_k with its set of P-positions identical to the set of N-positions of W^k.

1.1. **Some general results.** The set $\mathcal{R}_1$ has some frequently studied properties. Namely, the sequences $(\lfloor \phi n \rfloor)$ and $(\lfloor \phi^2 n \rfloor)$ are so-called *complementary sequences* of $\mathbb{N}$, e.g. [Fra82], that is they partition $\mathbb{N}$. (This follows from the well known 'Beatty's theorem' [Bea26].) In this paper we make use of a generalization of this concept—often used in the study of so called '(exact) covers by Beatty sequences' e.g. [Fra73, Gra73, Heg1].

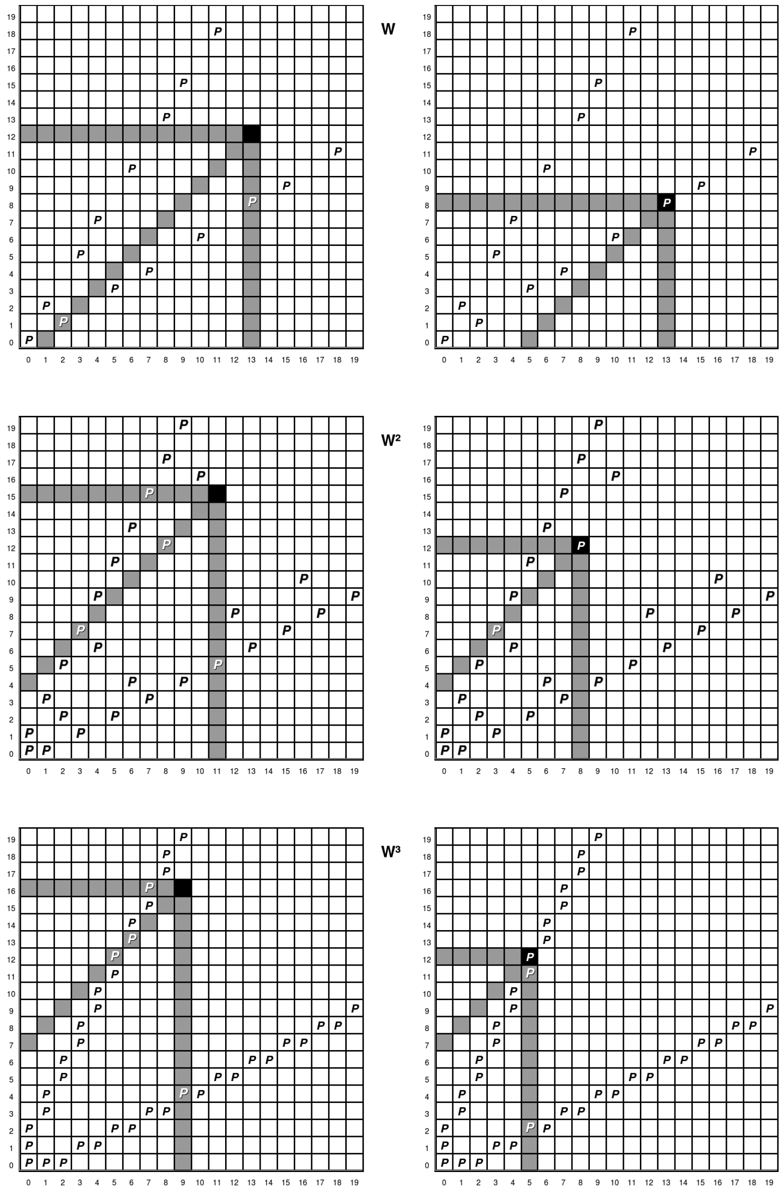

FIGURE 1. The two figures at the top illustrate options of two instances of Wythoff Nim together with its initial P-positions. The middle and lower couples of figures represent W^2 and W^3 respectively. For example in the middle left figure the 'gray' shaded positions are the options of the 'black' N-position $(11, 15)$. This position is N since, by rule of game, only one of the two P-positions in its set of options can be forbidden. In contrast, the position $(8, 12)$ is P (middle-right) since there is precisely one single P-position in its set of options. It can (and will) be forbidden.

Definition 5. *Let $p \in \mathbb{N}$. Suppose that A is a set of a finite number of sequences of non-negative integers. Then A is a* p-cover *(*cover *if $p = 1$) of another set, say $S \subset \mathbb{N}_0$, if, for each $x \in S$, the total number, $\xi(A, S, x)$, of occurrences of x, in the sequences of A, exceeds or equals p. Further, A is an* exact *p-cover of S if, for all x, $\xi(A, S, x) = p$.*

The special case of $S = \mathbb{N}$, $\#A = 2$ and $p = 1$ in this definition is 'complementarity'. For general p and with $\#A = 2$ the term p-complementarity is used in [Lar1].

Before we move on to the proof of Theorem 1.1, let us give some basic results valid for general W^k.

Proposition 1.2. *Let $k \in \mathbb{N}$ and define $\{\{a_i, b_i\} \mid i \in \mathbb{N}_0\} = \mathcal{P}_k$, where, for all i, $a_i \leq b_i$ and the ordered pairs (a_i, b_i) are in lexicographic order, that is (a_i) is non-decreasing and $a_i = a_j$ together with $i < j$ imply $b_i < b_j$. Then,*

(i) the 0^{th} column contains precisely k P-positions, namely

$$(0, 0), (0, 1), \ldots, (0, k - 1),$$

(ii) the set $\{(a_i), (b_i) \mid i \geq k\}$ is an exact k-cover of $\mathbb{N}$,

(iii) for all $d \in \mathbb{N}_0$, $\#\{i \in \mathbb{N}_0 \mid b_i - a_i = d\} \leq k$.

Proof. The case $k = 1$ follows from well known results on Wythoff Nim [Wyt07]. Hence, let $k > 1$. The item (i) is obvious (see also (2)). For (ii) suppose that there is a least $x' \in \mathbb{N}$ such that

$$r = \#(\{i \mid a_i = x'\} \cup \{i \mid b_i = x'\}) \neq k.$$

Clearly, by the blocking rule, this forces $r < k$ for otherwise there must trivially exist a non-blocked Nim-type move $\boldsymbol{x} \to \boldsymbol{y}$, where both $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{P}_k$. Suppose that y is the largest integer such that $(x', y) \in \mathcal{P}_k$. Then, by the blocking rule, for all integers

$$z > y, \tag{1}$$

there must exist a P-position in the set of horizontal and diagonal options of (x', z). (For otherwise all P-positions in the set of options of (x', z) could be blocked off.) But, by assumption, the total number of P-positions in the columns $0, 1, \ldots, x' - 1$ is precisely $k(x' - 1)$ and each such position is an option of precisely two positions in column x', which contradicts (1). Item (iii) is obvious by Definition 2. □

Notation 3. *A position (of W^k) is* terminal *if all options may be blocked off by the previous player.*

A player who moves to a terminal position may, by Definition 2, be declared the winner. Let $k \in \mathbb{N}$. The terminal positions of W^k are given by the following result. We omit the elementary proof.

Proposition 1.3. *Let $k \in \mathbb{N}$. The set of terminal positions of W^k is precisely*

$$\mathcal{T}(k) := \{\{x, y\} \mid x \leq y < k - 2x, x, y \in \mathbb{N}_0\}. \tag{2}$$

The set $\mathcal{T}(k)$ *is a* lower ideal, *that is* $(x,y) \in \mathcal{T}(k)$ *implies* $(x-i, y-j) \in \mathcal{T}(k)$, *for all* $i \in \{0,1,\dots,x\}$ *and all* $j \in \{0,1,\dots,y\}$. *The number of positions in this set is*

$$\#\mathcal{T}(k) := \begin{cases} 3(m+1)^2 - 2(m+1) & \textit{if } k = 3m+1, \\ 3(m+1)^2 & \textit{if } k = 3m+2, \\ 3(m+1)^2 + 2(m+1) & \textit{if } k = 3(m+1), \end{cases}$$

$m \in \mathbb{N}_0$.

In particular, the set of terminal positions of W^2 and W^3 are $\mathcal{T}(2) = \{(0,0),\{0,1\}\}$ ($\#\mathcal{T}(2) = 3$) and $\mathcal{T}^3 = \{(0,0),\{0,1\},\{0,2\}\}$ ($\#\mathcal{T}(3) = 5$) respectively.

2. Proof of the main result

Given a blocking parameter $k = 2$ or 3 and a position (x,y), we want to count the total number of options contained in our candidate set of P-positions $\mathcal{R}_2$ or $\mathcal{R}_3$ respectively. Then we may derive the value of (x,y) as follows. The previous player will win if and only if the total number of options in the candidate set is strictly less than k. With this plan in mind, let us define some functions, counting the number of options in some specific 'candidate set' and of the specific types, (v), (d) and (h) respectively.

Definition 6. *Let* $(x,y) \in \mathcal{B}$. *Given a set* $S \subset \mathcal{B}$, *let us define*

$$\begin{aligned} v_{x,y} = v_{x,y}(S) &:= \#(\{(w,y) \mid x > w \geq 0\} \cap S), \\ d_{x,y} = d_{x,y}(S) &:= \#(\{(w,z) \mid x - w = y - z > 0\} \cap S), \\ h_{x,y} = h_{x,y}(S) &:= \#(\{(x,z) \mid y > z \geq 0\} \cap S), \\ f_{x,y} = f_{x,y}(S) &:= d_{x,y} + v_{x,y} + h_{x,y}, \end{aligned}$$

$w, z \in \mathbb{N}_0$.

Notation 4. *We use the notation* $(x_1,x_2) \to (y_1,y_2)$ *if there is a Wythoff Nim (Definition 1) type move from* (x_1,x_2) *to* (y_1,y_2).

2.1. Proof of Theorem 1.1.

With notation as in Definition 4 and 6, put $S = \mathcal{R}_2$ and let $k = 2$. Hence, we consider the game W^2. Then, by the blocking rules in Definition 2, each P-position has the property that at most one of its options is P and each N-position has the property that at least two P-position are in its set of options. Thus, the theorem holds if we can prove that the value of $(x,y) \in \mathcal{B}$ is P if and only if $f_{x,y}(\mathcal{R}_2) \leq 1$. Hence notice that (see also (2))

- $f_{0,0} < f_{0,1} = 1$ and $(0,0), \{0,1\}$ are P,
- $x \geq 2$, $f_{0,x} \geq 2$ and $\{0,x\}$ is N,
- $f_{1,1} = 3$ and $(1,1)$ is N,
- $f_{1,2} = 2$ and $\{1,2\}$ is N.

Further, the 'least' P-position which is not terminal is $(2,2)$, namely $f_{2,2} = 1$ since, by the above items, the only option which is a P-position is $(0,0)$.

We divide the rest of the proof of the strategy of W^2 into two 'classes' depending on whether $(x,y) \in \mathcal{B}$ belongs to $\mathcal{R}_2$ or not.

Suppose that $(x,y)\in\mathcal{R}_2$. That is, we have to prove that $f_{x,y}(\mathcal{R}_2)\le 1$. We are done with the cases $(x,y)=(0,0),(0,1)$ and $(2,2)$. We may assume that $1\le x\le y$.

Case 1: Suppose that $y=2x+1$. Then, we claim that $h_{x,y}=0$, $d_{x,y}=0$ and $v_{x,y}=1$.

Proof. The horizontal options of $(x,2x+1)$ are of the form $(z,2x+1)$ with $z<x$. But all positions (r,s) in $\mathcal{R}_2$ satisfy

$$s\le 2r+1. \tag{3}$$

This gives $h_{x,y}=0$.

The diagonal options are of the form $(z,x+z+1)$, with $0\le z<x$. Again, by (3), this gives $d_{x,y}=0$.

For the vertical options, if $x\le 2$, we are done, so suppose $x>2$. Then, we may use that $\{(2\lfloor\phi n\rfloor+2)_{n\in\mathbb{N}},(2\lfloor\phi^2 n\rfloor+2)_{n\in\mathbb{N}},(2n+1)_{n\in\mathbb{N}}\}$ is an exact cover of $\{3,4,5,\ldots\}$.

Namely, if $x:=2z+1$ is odd, we have that

$$y=2x+1>x=2z+1>z,$$

so that $(x,y)\to(2z+1,z)\in\mathcal{R}_2$. Since x is odd, any vertical option in $\mathcal{R}_2$ has to be of this form.

If, on the other hand, $x:=2z\ge 2$ is even, then, since (by [Wyt07]) $(\lfloor\phi n\rfloor)_{n\in\mathbb{N}}$ and $(\lfloor\phi^2 n\rfloor)_{n\in\mathbb{N}}$ are complementary, there is precisely one n such that either $z=\lfloor\phi n\rfloor+1$ or $z=\lfloor\phi^2 n\rfloor+1$. For the first case

$$\begin{aligned}(x,2x+1)&=(2z,4z+1)\\&=(2\lfloor\phi n\rfloor+2,4\lfloor\phi^2 n\rfloor+3)\to(2\lfloor\phi n\rfloor+2,2\lfloor\phi^2 n\rfloor+2)\in\mathcal{R}_2.\end{aligned}$$

The second case is similar. But, since x is even, any option in $\mathcal{R}_2$ has to be precisely of one of these forms. We may conclude that $v_{x,y}=1$.

Case 2: Suppose that $x=2\lfloor\phi n\rfloor+2$ and $y=2\lfloor\phi^2 n\rfloor+2$, for some $n\in\mathbb{N}_0$. Then, we claim that $d_{x,y}=1$ and $v_{x,y}=h_{x,y}=0$.

Proof. If $n=0$, we are done, so suppose that $n>0$. We have that

$$(2\lfloor\phi n\rfloor+2,2\lfloor\phi n+n\rfloor+2)-(2n-1,4n-1)=(2\lfloor\phi n\rfloor-2n+3,2\lfloor\phi n\rfloor-2n+3)$$

is a diagonal move in Wythoff Nim (and where the '-'sign denotes vector subtraction). This gives $d_{x,y}\ge 1$. We may partition the differences of the coordinates of the positions in $\mathcal{R}_2$ into two sequences,

$$((2n+1)-n)_{n\in\mathbb{N}_0}=(n)_{n\in\mathbb{N}}$$

and

$$(2\lfloor\phi^2 n\rfloor+2-(2\lfloor\phi n\rfloor+2))_{n\in\mathbb{N}_0}=(2n)_{n\in\mathbb{N}}$$

respectively. These sequences are strictly increasing, which gives $d_{x,y}=1$.

For the second part we may apply the same argument as in Case 1, but in the other direction. Namely, $2x+1\ge 4\lfloor\phi n\rfloor+3>2\lfloor\phi^2 n\rfloor+2>2\lfloor\phi n\rfloor+2$, which implies that all nim-type options belong to the set $\mathcal{B}\setminus\mathcal{R}_2$.

We are done with the first class. Hence assume that $(x, y) \notin \mathcal{R}_2$. That is, we have to prove that $f_{x,y}(\mathcal{R}_2) \geq 2$.

Case 3: Suppose $y > 2x + 1$. Then we claim that $v_{x,y} = 2$, $h_{x,y} = 0$ and $d_{x,y} = 0$.

Proof. By the first argument in Case 1, the latter two claims are obvious. Notice that the set of sequences $\{(n)_{n\in\mathbb{N}},\ (2n+1)_{n\in\mathbb{N}_0},\ (2\lfloor\phi n\rfloor + 2)_{n\in\mathbb{N}_0},\ (2\lfloor\phi^2 n\rfloor + 2)_{n\in\mathbb{N}_0}\}$ constitute an exact 2-cover of $\mathbb{N}$. This gives $v_{x,y} = 2$.

Case 4: Suppose $0 < x \leq y < 2x + 1$. Then, we claim that either

(i) $d_{x,y} = 1$ and $h_{x,y} + v_{x,y} \geq 1$, or
(ii) $d_{x,y} = 2$.

Proof. We consider three cases.

(a) $y > \phi x$,
(b) $y < \phi x$ and $y - x$ even,
(c) $y < \phi x$ and $y - x$ odd.

In case (a), $v_{x,y} = 1$ is verified as in Case 1. For $d_{x,y} = 1$, it suffices to demonstrate that $(x, y) - (z, 2z+1) = (x - z, y - 2z - 1)$, is a legal diagonal move for some $z \in \mathbb{N}_0$. Thus, it suffices to prove that $x - z = y - 2z - 1$ holds together with $0 < z < x$ and $2z + 1 < y$. But this follows since the definition of y implies $z + 1 = y - x \leq 2x - x = x$.

In case (b) we get $d_{x,y} = 2$ by $(2\lfloor\phi^2 n\rfloor - 2\lfloor\phi n\rfloor)_{n\in\mathbb{N}} = (2n)$ and an analog reasoning as in the latter part of (a). (Hence this is (ii).)

In case (c) we may again use the latter argument in (a), but, for parity reasons, there are no diagonal options of the first type in (b), so we need to return to case (i) and thus verify that $h_{x,y} + v_{x,y} \geq 1$. Since $y - x$ is odd we get that precisely one of x or y must be of the form $2z+1$, $z \in \mathbb{N}_0$. Suppose that $z < x = 2z + 1 < y$. Then $(x, y) \to (2z + 1, z)$ gives $v_{x,y} \geq 1$. If, on the other hand, $x \leq y = 2z + 1 < \phi x$, then $(x, y) \to (z, 2z + 1)$ is legal since $z < \frac{\phi x - 1}{2} < x$, which gives $h_{x,y} \geq 1$.

We are done with W^2's part of the proof. Therefore, let $S = \mathcal{R}_3$, $k = 3$ and consider the game W^3. Then one needs to prove that $(x, y) \in \mathcal{R}_3$, $x \leq y$, if and only if $f_{x,y}(\mathcal{R}_3) \leq 2$. Suppose that $(x, y) \in \mathcal{R}_3$ with $x \leq y$. Then we claim that $d_{x,y} \leq 1$, $h_{x,y} = 0$ and $v_{x,y} \leq d_{x,y} + 1$. Otherwise, if $(x, y) \notin \mathcal{R}_3$ and $y > 2x + 2$, then we claim that $v_{x,y} = 3$, or, if $y < 2x + 1$, then we claim that $h_{x,y} = v_{x,y} = 1$ and $d_{x,y} \geq 1$. Each case is almost immediate by definition of $\mathcal{R}_3$ and Figure 1, so we omit further details. $\square$

3. Discussion

One obvious direction of future research is to try and classify the P-positions of the games W^k, $k \geq 4$. Let $k \in \mathbb{N}$ and, as in Proposition 1.2, let $\{\{a_i, b_i\} \mid i \in \mathbb{N}_0\} = \mathcal{P}_k$ denote the set of P-positions of W^k (with, for all i, $b_i \geq a_i$). Then we ask if

$$\lim_{i\to\infty} \frac{b_i}{a_i} \tag{4}$$

exists. If not, then we wonder if the set of aggregation points of $\lim_{i\to\infty}\frac{b_i}{a_i}$ is finite. More precisely: For some (least) $2 \le l = l(k) \in \mathbb{N}$, does there exist l sequences t^j, $j \in \{1, 2, \dots, l\}$, such that pairwise distinct asymptotic limits

$$\lim_{i\to\infty} \frac{b_{t_i^j}}{a_{t_i^j}} \tag{5}$$

exist? We conjecture that, for each $k \in \{4, 5, \dots, 20\}$ in Table 1, $l(k)$ is given by the number of entries in row k.

In [Lar2] another generalization of Wythoff Nim is studied, namely the family of Generalized Diagonal Wythoff Nim games and a so-called *split* of sequences of ordered pairs is defined. In particular a sequence of pairs $((a_i, b_i))$ is said to *split* if (4) is not satisfied but (5) is (for some $l \ge 2$). In that paper one conjectures quite remarkable asymptotic 'splits' of P-positions for certain games—supported by numerous computer simulations and figures. However, the only proof of a 'splitting' of P-positions given in that paper is the much weaker statement that (4) is not satisfied, and it is only given for one particular game called $(1, 2)$GDWN—a game which extends the diagonal options of Wythoff Nim and also allows moves of the types $(x, y) \to (x-i, y-2i)$, $x \ge i > 0, y \ge 2i > 0$ and $(x, y) \to (x-2i, y-i)$, $x \ge 2i > 0, y \ge i > 0$.

Remark 1. *In this paper we have proved that the 'upper' (above the main diagonal) P-positions of* W^2 *split. I am not aware of any other such result, of a split of 'the upper' P-positions of an impartial game, in particular not on a variation of Wythoff Nim. (if we drop the 'upper' condition then one may obviously regard the P-positions of, for example, Wythoff Nim as a splitting sequence, see* [Lar2]*).*

At the end of this section, we provide tables of the first few P-positions for W^4, W^5 and W^6 respectively. As an appetizer for future research on Blocking Wythoff Nim games, let us motivate the conjectured asymptote of (4) in Table 1 for the case W^4, that is that $\lim_{i\to\infty}\frac{b_i}{a_i}$ exists and equals $\sqrt{2}+1$. If (4) holds, with $\lim a_i/i = \alpha$ and $\lim b_i/i = \beta$ real numbers, then, by Proposition 1.2, also $\alpha^{-1} + \beta^{-1} = 4$ holds. Also, by Table 2, one hypothesis is that $\delta_n = b_n - a_n = n/2 + O(1)$, where $O(1)$ denotes some bounded function. This gives $\beta - \alpha = 1/2$ and so, by elementary algebra we get

$$\lim_{i\to\infty} \frac{b_i}{a_i} = \frac{\beta}{\alpha} = \sqrt{2} + 1.$$

3.1. **Comply- versus blocking-games.** Let $k \in \mathbb{N}$ and let G_k denote the following 'comply'-variation of any impartial game G. The previous player is requested to propose at least k of the options of G as allowed next-player options in G_k (and these are all options). After the next player has moved, this 'comply-maneuver' is forgotten and has no further impact on the game. The player who is unable to propose at least k next-player options is the loser[1]. It is not hard to check that this game has the 'reverse' strategy of G^k, where G^k is defined in analogy with W^k (in Definition 2 exchange Wythoff

[1] Our 'comply' games provide a subtle variation to those in [SmSt02].

Nim for G). By this we mean that the set of P-positions of G_k is precisely the set of N-positions of G^k.

Thus, as an example, let Nim_1 denote the comply-variation of Nim where the previous player has to propose at least one Nim option. In this game the empty pile is N (the previous player loses because he cannot propose any option). Each non-empty pile is P, since the previous player will propose the empty pile as the only available option for the next player. Recall that the only P-position of Nim (without blocking maneuver) is the empty pile.

This discussion motivates why we, in the definition of W^k (Definition 2), let the previous player forbid $k-1$, rather than k options. To propose at least k options is the 'complement' of forbidding fewer than k options—and it is not a big surprise that the set of P-positions of W^k are 'complementary' to those of W_k. (Another more 'algorithmic' way of thinking of this choice of notation is that (the position of) W^k a priory belongs to the set of forbidden options.)

Other blocking maneuvers on Wythoff Nim have been studied in the literature. Let $k \in \mathbb{N}$. In the game of k-blocking Wythoff Nim [HeLa06, Lar09, FrPe] the blocking maneuver only constrains moves of type (d) in Definition 1. Otherwise the rules are the same as in this paper. For the purpose of this section, denote this game by $\mathrm{W}^k\mathrm{N}$. In another variation, the game of Wythoff k-blocking Nim [Lar1], the blocking maneuver only constrains moves of type (h) or (v). Denote this game by WN^k. (Both these game families are actually defined, and solved, as restrictions of m-Wythoff Nim, [Fra82]).

The above discussion leads us to the following round up of this paper. Namely, let us attempt to define the corresponding 'comply rules' of $\mathrm{W}^k\mathrm{N}$ and WN^k, that is we want to find rules of games $\mathrm{W}_k\mathrm{N}$ and WN_k such that the P-positions of WN^k correspond precisely to the N-positions of WN_k and the P-positions of $\mathrm{W}^k\mathrm{N}$ correspond precisely to the N-positions of $\mathrm{W}_k\mathrm{N}$. We claim that the correct comply maneuver of $\mathrm{W}_k\mathrm{N}$ is: Propose at least k options of the type (d) *or* at least one Nim-type, (h) or (v), option. Similarly, for the game WN_k the comply rules are: Propose at least k Nim-type options *or* at least one option of type (d). Otherwise the rules are as in W_k.

Why are these the correct rules? Suppose, for example, that (x, y) is P in $\mathrm{W}^k\mathrm{N}$, say. Then there are at most $k-1$ (d) type P-positions and no Nim-type P-position at all in the set of options of (x, y). We have to demonstrate that (x, y) is N in $\mathrm{W}_k\mathrm{N}$, that is that the previous player cannot propose k (d)-type positions, all of them N, neither can he propose a single Nim-type N-position. But, by symmetry, this claim follows from a straightforward inductive argument, assuming complementarity of the P-positions in the two games on the $< x$ indexed columns. We omit further details. I could think of a few ways to continue this discussion, but it is nicer to leave some of this territory wide open in favor of others ideas.

$k=$	$l=1$	$l=2$	$l=3$	$l=4$
1	ϕ			
2	ϕ	2		
3	2			
4	$1+\sqrt{2}$			
5	1.476	2.5		
6	1.28	2.0	2.5	
7	2.5			
8	2.0	2.5		
9	1.34	2.5		
10	1.59	2.0	2.5	
11	2.5			
12	1.74	2.5		
13	1.2	2.5		
14	2.5			
15	$1+\sqrt{2}$	2.58		
16	1.426	2.5	2.6	
17	1.12	2.0	2.5	2.6
18	2.35	2.5	2.6	
19	1.88	2.5	2.6	
20	1.28	2.5	2.6	

TABLE 1. The entries in this table are the estimated/conjectured quotients $\lim_{i\to\infty} \frac{b_{t_i^j}}{a_{t_i^j}}$ for $j \in \{1,2,\ldots,l\}$ and the respective game W^k. The cases $k=2,3$ are resolved in Theorem 1.1 and $k=1$ is Wythoff Nim, $\phi = \frac{\sqrt{5}+1}{2}$.

n	a_n	b_n	δ_n	n	a_n	b_n	δ_n	n	a_n	b_n	δ_n
0	0	0	0	30	10	25	15	60	20	50	30
1	0	1	1	31	10	26	16	61	21	51	30
2	0	2	2	32	11	26	15	62	21	52	31
3	0	3	3	33	11	27	16	63	22	53	31
4	1	1	0	34	11	28	17	64	22	54	32
5	1	4	3	35	12	29	17	65	22	55	33
6	1	5	4	36	12	30	18	66	23	55	32
7	2	3	1	37	12	31	19	67	23	56	33
8	2	6	4	38	13	31	18	68	23	57	34
9	2	7	5	39	13	32	19	69	24	58	34
10	3	8	5	40	13	33	20	70	24	59	35
11	3	9	6	41	14	34	20	71	24	60	36
12	4	6	2	42	14	35	21	72	25	60	35
13	4	10	6	43	14	36	22	73	25	61	36
14	4	11	7	44	15	37	22	74	25	62	37
15	5	12	7	45	15	38	23	75	26	63	37
16	5	13	8	46	16	37	21	76	26	64	38
17	5	14	9	47	16	39	23	77	27	65	38
18	6	15	9	48	16	40	24	78	27	66	39
19	6	16	10	49	17	41	24	79	27	67	40
20	7	15	8	50	17	42	25	80	28	67	39
21	7	17	10	51	17	43	26	81	28	68	40
22	7	18	11	52	18	43	25	82	28	69	41
23	8	19	11	53	18	44	26	83	29	70	41
24	8	20	12	54	18	45	27	84	29	71	42
25	8	21	13	55	19	46	27	85	29	72	43
26	9	21	12	56	19	47	28	86	30	72	42
27	9	22	13	57	19	48	29	87	30	73	43
28	9	23	14	58	20	48	28	88	30	74	44
29	10	24	14	59	20	49	29	89	31	75	44

TABLE 2. The first few P-positions of W^4, $\{a_n, b_n\}$, and the corresponding differences $\delta_n := b_n - a_n$.

n	a_n	b_n	δ_n	n	a_n	b_n	δ_n	n	a_n	b_n	δ_n
0	0	0	0	30	8	21	13	60	17	24	7
1	0	1	1	31	8	22	14	61	17	43	26
2	0	2	2	32	8	23	15	62	17	44	27
3	0	3	3	33	9	23	14	63	17	45	28
4	0	4	4	34	9	24	15	64	18	46	28
5	1	1	0	35	9	25	16	65	18	47	29
6	1	2	1	36	10	14	4	66	18	48	30
7	1	5	4	37	10	26	16	67	19	25	6
8	1	6	5	38	10	27	17	68	19	48	29
9	2	4	2	39	10	28	18	69	19	49	30
10	2	7	5	40	11	16	5	70	19	50	31
11	2	8	6	41	11	28	17	71	20	29	9
12	3	3	0	42	11	29	18	72	20	51	31
13	3	6	3	43	11	30	19	73	20	52	32
14	3	9	6	44	12	15	3	74	20	53	33
15	3	10	7	45	12	31	19	75	21	31	10
16	4	11	7	46	12	32	20	76	21	53	32
17	4	12	8	47	12	33	21	77	21	54	33
18	4	13	9	48	13	33	20	78	21	55	34
19	5	7	2	49	13	34	21	79	22	30	8
20	5	13	8	50	13	35	22	80	22	56	34
21	5	14	9	51	14	36	22	81	22	57	35
22	5	15	10	52	14	37	23	82	22	58	36
23	6	16	10	53	14	38	24	83	23	58	35
24	6	17	11	54	15	38	23	84	23	59	36
25	6	18	12	55	15	39	24	85	23	60	37
26	7	18	11	56	15	40	25	86	24	61	37
27	7	19	12	57	16	41	25	87	24	62	38
28	7	20	13	58	16	42	26	88	24	63	39
29	8	9	1	59	16	43	27	89	25	63	38

TABLE 3. The first few P-positions of W^5, $\{a_n, b_n\}$, and the corresponding differences $\delta_n := b_n - a_n$.

n	a_n	b_n	δ_n	n	a_n	b_n	δ_n	n	a_n	b_n	δ_n
0	0	0	0	30	6	19	13	60	13	36	23
1	0	1	1	31	7	15	8	61	14	27	13
2	0	2	2	32	7	19	12	62	14	37	23
3	0	3	3	33	7	20	13	63	14	38	24
4	0	4	4	34	7	21	14	64	14	39	25
5	0	5	5	35	8	9	1	65	15	30	15
6	1	1	0	36	8	22	14	66	15	39	24
7	1	2	1	37	8	23	15	67	15	40	25
8	1	3	2	38	8	24	16	68	15	41	26
9	1	6	5	39	9	16	7	69	16	32	16
10	1	7	6	40	9	24	15	70	16	42	26
11	2	4	2	41	9	25	16	71	16	43	27
12	2	5	3	42	9	26	17	72	16	44	28
13	2	8	6	43	10	20	10	73	17	18	1
14	2	9	7	44	10	27	17	74	17	31	14
15	3	6	3	45	10	28	18	75	17	44	27
16	3	7	4	46	10	29	19	76	17	45	28
17	3	10	7	47	11	11	0	77	17	46	29
18	3	11	8	48	11	22	11	78	18	35	17
19	4	8	4	49	11	29	18	79	18	47	29
20	4	12	8	50	11	30	19	80	18	48	30
21	4	13	9	51	11	31	20	81	18	49	31
22	4	14	10	52	12	21	9	82	19	37	18
23	5	10	5	53	12	32	20	83	19	49	30
24	5	14	9	54	12	33	21	84	19	50	31
25	5	15	10	55	12	34	22	85	19	51	32
26	5	16	11	56	13	13	0	86	20	40	20
27	6	12	6	57	13	25	12	87	20	52	32
28	6	17	11	58	13	34	21	88	20	53	33
29	6	18	12	59	13	35	22	89	20	54	34

TABLE 4. The first few P-positions of W^6, $\{a_n, b_n\}$, and the corresponding differences $\delta_n := b_n - a_n$.

References

[ANW07] M. H. Albert, R. J. Nowakowski, D. Wolfe *Lessons in Play: In Introduction to Combinatorial Game Theory.* A K Peters Ltd.(2007).

[BCG82] E. R. Berlekamp, J. H. Conway, R.K. Guy, *Winning ways*, **1-2** Academic Press, London (1982). Second edition, **1-4**. A. K. Peters, Wellesley/MA (2001/03/03/04).

[Bea26] S. Beatty, Problem 3173, *Amer. Math. Monthly*, **33** (1926) 159.

[Con76] J.H.Conway, *On Numbers and Games*, Academic Press, London (1976). Second edition, A.K.Peters, Wellesley/MA (2001).

[Fra73] A.S. Fraenkel, Complementing and exactly covering sequences, J. Combinatorial Theory Ser. A **14** (1973), 8-20.

[Fra82] A. S. Fraenkel, How to beat your Wythoff games' opponent on three fronts, *Amer. Math. Monthly***89** (1982) 353-361.

[Fra04] A. S. Fraenkel, Complexity, appeal and challenges of combinatorial games. *Theoret. Comp. Sci.*, **313** (2004) 393-415.

[FrPe] A.S. Fraenkel, U. Peled, Harnessing the Unwieldy MEX Function, preprint, http://www.wisdom.weizmann.ac.il/ fraenkel/Papers/ Harnessing.The.Unwieldy.MEX.Function_2.pdf.

[Gra73] R.L. Graham, Covering the positive integers by disjoint sets of the form $\lfloor n\alpha + \beta \rfloor : n = 1, 2, ...,$ *J. Combinatorial Theory Ser.* A **15** (1973), 354-358.

[Heg1] On m-covering families of Beatty sequences with irrational moduli, preprint.

[HeLa06] P. Hegarty and U. Larsson, Permutations of the natural numbers with prescribed difference multisets, *Integers* **6** (2006), Paper A3, 25pp.

[Lar09] U. Larsson, 2-pile Nim with a Restricted Number of Move-size Imitations, *Integers* **9** (2009), Paper G4, pp 671-690.

[Lar1] U. Larsson, Restrictions of m-Wythoff Nim and p-complementary Beatty sequences, to appear in *Games of no Chance 4*.

[Lar2] Urban Larsson, A Generalized Diagonal Wythoff Nim, to appear in *Integers*.

[SmSt02] F. Smith and P. Stănică, Comply/Constrain Games or Games with a Muller Twist, *Integers*, **2**, (2002).

[Wyt07] W.A.Wythoff. A modification of the game of Nim. *Nieuw Arch. Wisk.* **7** (1907) 199-202.

E-mail address: `urban.larsson@chalmers.se`

Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, Göteborg, Sweden