Some Generalisations to the Combinatorial Game of Wythoff Nim

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What is a combinatorial game?

The usual definition is a game in which:
1) There are two players moving alternately;
2) There are no chance devises and both players have perfect information;
3) The rules are such that the game must eventually end; and
4) There are no draws (this rule can sometimes be adjusted to allow draws) and the winner is determined by who moves last;

The (general) theory of combinatorial games was developed by Conway in the 1970:s. Each combinatorial game is in some sense equivalent to a (generalized) number.

Our games consist of a finite number of game positions. Given a game, to each position there is a finite number of options (moves). A position A is a (direct) *follower* of a position B if there is a move from B to A. Positions from which the $Previous\ player$ (P-player) can win whatever move his opponent will make are called

$$P - positions,$$

and those from which the $Next\ player$ (N-player) can always win is called

$$N - positions.$$

A final $P-$position is called a

$$sink.$$

This is where the game terminates. If the last player to move wins then we play under normal rules, otherwise we play a *misére* version.

Note that a P-position can only have N-positions as followers whereas for an N-position there is at least one P-position as a direct follower.

What is a (finite) impartial game?

A game is impartial if both players have the same options, and all options are impartial games themselves;

otherwise the game is partisan (like chess, checkers, go and hackenbush).

We will only deal with impartial games following the normal play rule where the last player to move wins. One of the most famous example is the game of Nim.

The game of Nim (Bouton 1902)

Given a positive number of piles, each pile with a positive number of coins, the two players move alternately. A player may remove any positive number of coins from a single pile. The first player unable to move loses the game.

Bouton discovered that the P-positions of this game can be determined by binary addition without carry such that if we have n piles with $x_1, \ldots, x_n$ coins, then the previous player will eventually win the game (given best play from both parts) iff $x_1 \oplus x_2 \ldots \oplus x_n = 0$. Here $\oplus$ denotes binary addition without carry, the XOR operator.

Note: One may create new Nim-like games by by introducing restrictions to the Nim options. We get an infinite class of so called (n-pile) *subtraction games*.

Notation: We denote a position in a game with n piles of sizes $x_1, \dots, x_n$ respectively as $(x_1, \dots, x_n)$. We follow the convention that different permutations of the x_i constitutes equivalent game positions. For example in two-pile Nim $(2, 3)$ is equivalent to $(3, 2)$.

There is a complete theory of impartial games given normal play rules (not misére play) developed independently by Sprague (1936) and Grundy (1939).

Each position of an impartial game is equivalent to a Nim heap with a *Grundy value* $g(\cdot)$, defined recursively as the least natural number not obtained by any of its direct followers. A sink has Grundy value 0.

Suppose G and H are two impartial games, then the sum of G and H, denoted by $G \oplus H$, is also an impartial game, played as follows.

The next player makes a move in either G or H and leaves the other component-game untouched.

The sum $G \oplus H$ is also an impartial game. A position (x, y) of $G \oplus H$ is $P-$position iff $g(x) + g(y) = 0$.

What is a (move-size/pile-size) dynamic impartial game?

One-pile Nim with no additional rules is of course trivial to play. The first player always wins simply by removing all coins from the table. But we can construct new games by giving new restrictions to the rules of one-pile Nim.

Fibonacci Nim:
The number of coins that the next player may remove must not exceed twice the number of coins that the previous player removed in his last move. Also, the first player may not take the whole heap. Now we have a ”playable” game. Is there a winning strategy for this game?

Yes! The $P-$positions are heaps with a Fibonacci number $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ of coins, (rather: a heap should be a Fibonacci number and the movesize must be a non-Fibonacci number).

A winning strategy is derived from the Zeckendorf expansion of a natural number: In order to prevent the next player from moving to a fibonacci position it suffices to take away the least "Zeckendorf component" number of coins.

Example: For a Nim heap with $54 = 2 + 5 + 13 + 34$ coins you are safe if you remove 2 coins (but not if you remove 13 coins!).

In the article: ”One pile Nim with arbitrary move function”, (CiteSeer 2003), Holshouser and Reiter resolve the question of determining winning positions for one-pile counter pickup games with an arbitrary move-size function

$$F : \mathbf{N} \to \mathbf{N}$$

(for Fibonacci Nim, $F(x) = 2x$). They use generalized integer bases and nicely expands the idea presented for Fibonacci Nim.

What is a Muller twist?

We say that we put a *Muller twist* on a familiar combinatorial game by modifying that game so that each player's move of game pieces in the familiar game is followed by that players constraint, from a well defined set of constraints, on the next players move.

The origin of this concept is the game Quarto created by Blaise Muller. It was one of the five Mensa games of the year in 1993.

It is a board game for two players, played on a 4×4 board. There are 16 unique pieces, each of which has four attributes:

1. large or small;
2. red or blue (or some other pair of colors, such as light or dark stained wood);
3. square or circle;
4. hollow or solid.

Players take turns to choose a piece which *the other player* must then place on the board.

A player wins by placing their piece to make four pieces in a horizontal, vertical, or diagonal row, all of which have a common attribute (all small, all circular, etc).

Luc Goossens has shown in 1998 that, given best play from both sides, this game will end in a draw.

Altogether, there is not much known on blocking impartial games, the main contributors are Athur Holshouser and Harold Reiter, University of North Carolina Charlotte. They have recently submitted an article on the Sprague Grundy theory for "Blocking combinatorial games" (Integers).

As an easy example, let us look at two-pile Nim with a Muller twist, where the P-player blocks exactly one option. For clarity, notice that for example the option $(1, 1) \rightarrow (1, 0)$ is not equivalent to $(1, 1) \rightarrow (0, 1)$. So given $(1, 1)$, the P-player can not prevent the N-player from moving to the P-position $(0, 1)$. The position $(1, 1)$ is therefore the 'least' N-position in this game.

The P-positions are $(0, 0)$ together with the pairs $(a, a + 1)$ for $a \geq 0$. The N-positions are the complement, every pair of positive natural numbers (a, b) with $a \neq b - 1$.

What is a winning strategy if we adjust the rules so that we block exactly one position (rather than one option)?

It is easy to see that P-positions are precisely the set $\{(a, a), (a, a+1) \mid 0 \leq a \text{ even}\}$.

In 2003 Holshouser and Reiter, "Three Pile Nim with Move Blocking", analyses the winning positions for three-pile Nim with exactly one blocked option. They start the presentation by a comparison with three-pile Nim:

"Remarkably, this apparently more complicated game yields a strategy that does not require binary arithmetic."

The winning positions of this game are all positions of the form (a, a, a) where $a \geq 0$ together with the positions (a, b, c) such that $a + b + 1 = c$.

According to Harold Reiter, four-pile Nim with blocking is still "wide open" (at least yesterday). Three pile Nim with two blocked options also probably has an unknown winning strategy.

There are a few more contributions in the literature. All very young. Flammenkamp, Holshouser and Reiter have a specific one-pile Nim result in "Dynamic one-pile blocking Nim" (EJC 2003).

Another example is a result of *odd-or-even Nim*, due to F.Smith and P.Stanica (Integers 2002).

This game is n-pile Nim with the Muller twist: The previous player decides which parity of number of coins the next player must remove. The $P-$positions of this game is determined by:

Nimsum = 0 and the restriction is even;
Nimsum = 1 and the restriction is even; or
Nimsum = 0, all piles are even and the restriction is odd.

This result has been generalized by H.Gavel and P.Strimling in the article "Nim with a modular Muller Twist" (Integers 2004).

We return to the non-blocking version of Nim for a moment. The game of Nim played on only two piles is trivial. The P-positions consists of every position (k, k) with $k \in \mathbf{N}$.
We can play a much more interesting game by adding Nim's P-positions as options:

The game of Wythoff Nim (Wythoff 1907) is also known under the name "Corner the queen" in the west and apparently also in (ancient?) China under the name of Tsianshidsi. This game is a finite impartial 2-person game with rules as in the game of 2-pile Nim with one addition:

A player may at his turn take away the same positive number of coins from both piles.

One might want to think of the move rules as the options for a single queen on a chessboard with the restriction that on a move the distance to the left-down corner must always decrease.

We can recursively define the P-positions of this game by using the (minimal exclusive) *mex* definition: Suppose A is a (possibly empty) set of natural numbers. Then *mex*A is the least natural number in the complement of A.

Clearly $(0, 0)$ is the unique sink of the game of Wythoff Nim. Let us denote the P-positions with (a_n, b_n) for $n \in \mathbf{N}$. Then

$$a_n = \textit{mex}\{a_i, b_i \mid 0 \leq i < n\}$$

and $b_n = a_n + n$. The winning strategy follows easily from this, but unfortunately this algorithm has an exponential time complexity (in succinct input size).

There is a wellknown "polynomial time" winning strategy for the game of Wythoff Nim:

Proposition 1 *The* P*-positions of this game* $= \{(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor) \mid n \in \mathbf{N})\}$*, where* $\phi = \frac{1+\sqrt{5}}{2}$*, the golden ratio.*

These so called *Wythoff pairs* partition the natural numbers. This beautiful result is a special case of Beattys theorem (Beatty 1928):

Theorem 1 *Let r and s be positive irrational numbers such that*

$$\frac{1}{r} + \frac{1}{s} = 1.$$

Further, let

$$A = \{\lfloor nr \rfloor \mid n \in \mathbf{N}\}$$

and

$$B = \{\lfloor ns \rfloor \mid n \in \mathbf{N}\}.$$

Then $A \cap B = \emptyset$ and $A \cup B = \mathbf{N}$.

Let m be a positive integer. In the paper "How to beat your Wythoff Games opponent on three fronts" (1982) Aviezri S. Fraenkel, Weizman institute of Science Rehovat, generalises the game of Wythoff Nim to a family of 'less restrictive' games that we denote with m-Wythoff:

Given two piles of tokens, two players move alternately. The moves are of two types: a player may remove any positive number of tokens from a single pile, or he may take from both piles, say $k > 0$ from one and $l > 0$ from the other, provided that $\mid k - l \mid< m$.

It is easy to define these generalized Wythoff pairs with the *mex* definition, but what is more interesting is that the P-positions in m-Wythoff for $m \geq 2$ also constitute Beatty sequences:

Proposition 2 *The P-positions of this game* $= \{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor) \mid n \in \mathbf{N}\}$, *where*

$$\alpha = \frac{2 - m + \sqrt{m^2 + 4}}{2}$$

and $\beta = \alpha + m$*.*

Then β and α are irrational and satisfy $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, so Beattys theorem gives the corresponding partitioning of the natural numbers. The computation of winning positions can be done in polynomial time.

In a subsequent article ”New games related to old and new sequences” (Integers 2004), Fraenkel generalises these Wythoff games to dynamic games.

p-**Blocking** m-**Wythoff Nim** This is the generalization of the m-Wythoff game for which the P-positions are precisely the pairs

$$(n-1, \pi_{m,p}(n)-1)$$

for $n \geq 1$. The rules of the game are just as in the m-Wythoff game, with one vital exception:

Before each move is made, the previous player has a choice to block at most $p-1$ of the possible options where the next player would have removed an equal number of coins from each pile.

In other words: If the current configuration is (k, l), then before the next move is made, the previous player is allowed to choose up to $p-1$ distinct, positive integers $c_1, ..., c_{p-1} \leq \min\{k, l\}$ and declare that the next player may not move to any configuration $(k - c_i, l - c_i)$.

Beatty sequences and p-blocking Wythoff Nim

For $m = 1$ and any $p > 1$, one can show that it won't be possible to express the pairs $(n, \pi_{1,p}(n))$ as $(\lceil nr \rceil, \lceil ns \rceil)$ for any real r and s satisfying $\frac{1}{r} + \frac{1}{s} = 1$, and depending only on p. However, Hegarty has recently proved an asymptotic result (leaving the exact configuration of the P-positions of p-blocking Wythoff Nim as an open question):

Take

$$r = r_p = \frac{(2p-1) + \sqrt{4p^2+1}}{2p}$$

and $s = s_p = r_p + \frac{1}{p}$. Then $\frac{1}{r} + \frac{1}{s} = 1$ and with $\{k \mid \pi_{1,p}(k) > k\} = (a_k)$ we have that (with the $a_k : s$ listed in increasing order):

Theorem 4 *For all $k > 0$,*

$$|\ a_k - \lceil kr \rceil\ | \leq\ p - 1.$$

Let us return to p-blocking m-Wythoff Nim for a moment:

For the special case where $p \mid m$ we know that the P-positions are precisely the pairs (px_n, py_n), $(px_n + 1, py_n + 1) \ldots, (px_n + p - 1, py_n + p - 1)$, where (x_n, y_n) are the P-positions in the game of $(\frac{m}{p})$-Wythoff.

For any other m and p we conjecture that there is a constant $c = c_{m,p}$ depending only on m and p, such that b_k differs with at most c from $L = \frac{m+\sqrt{m^2+4p^2}}{2p}$.

There is an interesting question left to resolve: Given a symmetric multiset and 'it's greedy permutation'. Is there an 'analog' two-pile subtracion game with a winning strategy?

We have a proposition that aims to generalize the main theorem in A. Fraenkels article "New games related to old and new sequences", namely to put a pile-size dynamic Muller twist on Fraenkels pile-size dynamic games.

Fraenkel defines an infinite class of 2-pile subtraction games, where the differing amount that can be subtracted from both piles simultaneously, is a function f of the pile-sizes of (possibly) both the previous and the next player.

m-Wythoff is the special case $f = m$.

For each game, a minimal exclusive algorithm in analog to the one we presented in the context of Wythoff Nim, generates a pair of complementary sequences. The main result is a theorem giving necessary and sufficient conditions on f so that the sequences are P-player winning positions.

Our proposed extension:

Suppose two functions $f : \mathbf{N} \times \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ and $g : \mathbf{N} \times \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ are given. Henceforth ξ denotes a $\{0, 1\}$-valued function on the non-negative integers such that:

$$\xi(n) = 1$$

if and only if

$$\#\{i \in [0, n-1] \mid b_{n-1} - a_{n-1} = b_i - a_i\} =$$

$$g(a_{i^*}, b_{i^*}, a_{n-1}),$$

where i^* denotes the least index i such that $b_{n-1} - a_{n-1} = b_i - a_i$.

We may recursively define two infinite sequences of integers (a_i) and (b_i) that together partition the positive integers. First, put $a_0 = b_0 = 0$. For $n > 0$, let

$$a_n = \text{mex}\{a_i, b_i \mid i \in [0, n-1]\},$$

where *mex* has the usual meaning, and let

$$b_n = a_n + b_{n-1} - a_{n-1} + \xi(n) f(a_{n-1}, b_{n-1}, a_n).$$

We want to give sufficient (and necessary) conditions to our move functions f and g for the sequence (a_i, b_i) to constitute the P-positions. The options for the game, that we denote by (f, g)-Wythoff, are in analog to p-blocking m-Wythoff with m exchanged for f and p exchanged for g. We propose f and g such that:

The pile-size dynamic constraints:

1) *f-positivity,*

$$f(x, y, x') > 0$$

if $x \le y$ and $x < x'$;

2) *Increasing monotonicity*,

$$f(x, y, x') \le f(x, y, x'')$$

if $x < x' \le x''$ and $x \le y$;

3) g*-weighted semi-additivity,* for integers $0 \le m < n$,

$$\sum_{i=0}^{m} \xi(n-i) f(a_{n-i-1}, b_{n-i-1}, a_{n-i}) \ge$$

$$f(a_{n-m-1}, b_{n-m-1}, a_n),$$

where ξ is as above.

These constraints are in essence derived from A. Fraenkels article (remove the function ξ).

The blocking dynamic constraints:

1') *g-Positivity*,

$$g(x, y, x') > 0,$$

if $x < x'$ and $x \leq y$;

2') *Decreasing monotonicity*,

$$g(x, y, x') \geq g(x, y, x''),$$

if $x < x' < x''$ and $x \leq y$;

3') *Blocking monotonicity*,

$$g(x, y, x'') \leq g(x', y', x''),$$

if $x < x' < x''$ and $y' - x' = y - x$.

For each one of the conditions 1,2,3,1',2' and 3' one can show that without precisely this condition, there is a game that satisfies: there is an i such that either (a_i, b_i) is an N-position or (a_i, y) is a P-position and $y \neq b_i$ or (x, b_i) is a P-position and $x \neq a_i$.

Thank you!